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Model-based estimation of the stress-strain curve of metal strips
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ABSTRACT
The identification of the stress-strain curve of metal strips is a common task in the metals industry. As an alternative to commonly used tensile test machines, an inexpensive, model-based optical measurement method is presented. Particular importance was placed on the cost and usability of the method. The indirect approach computes the stress-strain curve based on a measured strip bending line. For the measurement, a metal strip is bent over a solid roll. A defined weight can be mounted at the end of the strip to control the local bending moment in the strip. The bending line of the strip is optically measured by a camera. The identification is carried out based on an optimization problem, where the quadratic error between the measured and the modeled strip bending line is minimized. Experimental results and measurements from a tensile test machine show a good agreement and thus verify the proposed identification method.

1. Introduction
Tensile testing is a well established part of materials testing for the determination of the stress-strain curve, which is an important material property. In the tensile test, standardized specimens with well-defined cross-sectional area are elongated until failure, cf. \cite{1–3}. During the test, the force acting on the sample and the strain are continuously measured to obtain the stress-strain curve. In \cite{4}, miniature tensile specimens, which are generally too small to be used in conventional tensile testing machines, are considered. The influence of specimen dimensions and strain measurement methods on tensile stress-strain curves are investigated. It is shown that the strain measurement is highly influenced by the specimen dimensions. As a result, an increase in uniform elongation and post-necking elongation can be observed for decreasing gauge length and increasing specimen thickness. An apparatus for material testing in high pressure hydrogen gas is discussed in \cite{5}. Tensile tests in 70 MPa hydrogen at room temperature were successfully performed. A lot of research was carried out on tensile testing of micromechanical materials, e.g. the tensile testing of polysilicon was examined in \cite{6}, where the advantages and drawbacks of several approaches for determining Young’s modulus are discussed. Extensions to nano-scale and meso-scale tensile testing of
materials can be found in [7,8]. As discussed in [8], (small-scale) tensile testing approaches with uniaxially applied load cause a nominally uniform stress and strain state in the specimen. The high-temperature tensile tester outlined in [7] allows measurements for temperatures up to 1200°C and strain rates between $10^{-5}$/s to $10^{-1}$/s at the meso-scale. High strain rate tensile testing of sheet materials is reported in [9], where a split Hopkinson pressure bar system (a specimen is positioned between the ends of two straight bars) and a load inversion device are used. Under certain assumptions about the specimen, parasitical oscillations in the measured stress-strain curve can be avoided. To increase the range of measurement of the stress-strain curve, the application of high-pressure torsion is proposed in [10]. This facilitates the deformation of the specimen without fracture and therefore enables flow stress measurement up to a strain of 10.0. The determination of the linear elastic modulus (Young’s modulus) from tensile tests of sheet metals by means of utilizing the deformation work is outlined in [11]. Multiple articles that deal with the determination of the stress-strain curve beyond necking can be found in the literature, see, e.g. [12–15]. In [13], the experimental use of a photographic system is proposed to observe necking during dynamic tensile tests performed with a split Hopkinson tension bar and the use of the Bridgman solutions [16]. The classical Bridgman approach [16] is concerned with the extraction of post-necking strain hardening data while performing a uniaxial tension test. An extension of the classical Bridgman approach, i.e. cylindrical tension samples, to thin sheet metals with non-symmetric and irregular necking morphologies, can be found in [14]. In [17], the identification of Young’s modulus of an aluminum bar is considered. The main focus lies on edge and shape detection by proposing a virtual image correlation method. However, the example problem shown in [17] is restricted to a linear elastic material behaviour. Moreover, image distortion is not considered. To sum up, a few disadvantages of current tensile tests are the following:

- the strain has to be measured accurately, e.g. by an extensometer, a strain gauge or laser interferometry
- the dimension of the specimen is typically prescribed by the machine,
- for tests at elevated temperatures, the specimen is typically heated and the tests are carried out in an environmental chamber [2].

The aim of this paper is to develop a method for the identification of the stress-strain curve of a metal strip with minimum effort and costs. Furthermore, the presented method allows for the identification of the plastic domain of the stress-strain curve and Poisson’s ratio, respectively, whereas only the estimation of Young’s modulus was considered in [17]. Many features and advantages of the tensile test methods cited in Section 1 are also offered by the proposed method. In particular, the use of various specimen sizes, measurements in a high pressure hydrogen gas atmosphere as well as in a high temperature environment are easily possible without substantial changes of the proposed experimental setup. For instance, the setup can be readily put into a furnace to carry out measurements at an elevated temperature. In fact, even creep-tests are possible by means of this setup.

For the mathematical modelling it is assumed that the material conforms to the Prandtl-Reuß equations. In order to carry out the identification procedure, a laboratory experiment, which is shown in Figure 1, was built.
The test rig consists of a roll with radius $R$ on which a steel strip with the length $L$ can be attached. The roll can be manually rotated to coil the strip and to control the angular position of the strip-roll contact point. Let $x$ define a position along the steel strip according to Figure 2. To change the local bending moment $m_b(x)$ in the strip, a defined weight can be mounted at the end of the strip. In strip sections with strip-roll contact, the longitudinal curvature $\kappa(x)$ of the strip is $1/(R + h/2)$ with the roll radius $R$ and the thickness $h$ of the strip. Then, the maximum strain in the strip is known in advance. Let $m_c(\kappa)$ denote the constitutive law between curvature $\kappa$ and bending moment. The local bending moment $m_b(x)$ follows from the angular momentum balance, see Figure 2. Clearly, $m_b(x) = m_c(\kappa(x))$ holds true, i.e. $m_b$ is a function of the local position and $m_c$ refers to the constitutive law and is thus a function of $\kappa$. Assuming that the elongation of the strip is negligible and that $h \ll L$, a bijective mathematical relation between the stress-strain curve $\sigma(\varepsilon)$ and the curvature-moment relation $m_c(\kappa)$ exists. Therefore, the stress-strain curve can be identified by means of the measured bending line, if the load conditions are known. During the experiment, the local bending load of the strip does not decrease (an unloading cycle does not occur) and thus the stress-strain curve is traversed with increasing strains. To

**Figure 1.** Test rig for new measurement method.

**Figure 2.** Geometry of the measurement device (left) and local coordinate system for the determination of the deflection of the strip (right).
ensure stable load conditions, i.e. to rule out snap-through effects, a monotonous constitutive law \( m_c(\kappa) \) is assumed in the relevant loading range. Basically, two approaches are possible for the identification:

(a) naive solution (no optimization-problem)
- identify the bending line \( y(x) \) by curve fitting
- compute the bending moment \( m_b(x) \) in the strip
- compute \( \kappa(x) \) by differentiation of \( y(x) \)
- use \( m_b(x) \) and \( \kappa(x) \) to get \( m_c(\kappa) \)
- compute \( \sigma(\varepsilon) \) by differentiation of \( m_c(\kappa) \) (see Sec. Appendix for details)

(b) suggested solution
- identify the bending line \( y(x) \) by curve fitting
- compute the bending moment \( m_b(x) \) in the strip
- parameterize \( \sigma(\varepsilon) \) (optimization parameters to be estimated)
- compute \( m_c(\kappa) \) by integration
- use \( m_c(\kappa) \) to get \( \kappa(x) \)
- compute \( y(x) \) by integration of \( \kappa(x) \)
- optimization-problem: minimize the quadratic error between the measured and the modelled strip bending line

A major drawback of variant a) is that the estimation of \( \sigma(\varepsilon) \) requires differentiations at two separate occasions. First, the bending line \( y(x) \) has to be differentiated twice with regard to \( x \) to compute \( \kappa(x) \). Second, \( dm_c(\kappa)/d\kappa \) is required in the computation of \( \sigma(\varepsilon) \) as shown in Sec. Appendix. Slight changes in the curve fit may lead to substantial changes in the curvature. The quantity \( dm_c(\kappa)/d\kappa \) is thus very sensitive to the curve fit and variant a) is therefore completely useless in practice. The suggested variant b) avoids derivatives at any time (integration is a numerically benign operation) and has proven to be very robust against the curve fit of the bending line. The main advantages of the proposed measurement method are the following:

- inexpensive,
- simple to use,
- no tensile testing machine necessary,
- dimension of the specimen is widely unrestricted,
- robust because material properties are averaged over a large specimen area,
- in principle also applicable for elevated temperatures.

This paper is structured as follows: In Sec. 2, image processing for measuring the deflection, in particular image distortion and edge detection is considered. Furthermore, the bending moment in the strip is computed. The derivation of the optimization problem for the identification of the stress-strain curve and Poisson’s ratio is described in Sec. 3. Finally, experimental results and measurements that validate the identification method are presented in Sec. 4.
2. Image processing and bending moment in the strip

This section deals with the computation of the bending moment in the strip. For this purpose, the bending deflection of the strip is captured by a CCD camera using a picture resolution of 4000 × 6000 pixels. Four lamps were used to improve the contrast.

2.1. Geometry of the measurement device

Figure 2 shows the geometry of the measurement device and the local coordinate system of the loaded strip. Let $\alpha$ be the angular position of the clamped end of the strip, $\alpha_1$ the angular position of the end of the strip-roll contact, $s$ the arc length of the strip, and $l = L - R(\alpha - \alpha_1)$ the length of the free strip section. Hence, a normalized strip length coordinate

$$\tilde{s} = \frac{s}{l} \in [0, L/l]$$  \hspace{1cm} (1)

can be introduced. The bending moment $m_b$ per unit strip width in the free strip section ($\tilde{s} \in [0, 1]$) caused by the dead load of the strip and the load force $F_s = m_s g$, with the mass of the extra load $m_s$ and the gravitational acceleration $g$, can be obtained as a function of the normalized strip length coordinate $\tilde{s}$ by

$$m_b(\tilde{s}) = \frac{1}{b} \left( F_s x(\tilde{s}) \right) + g \rho h \int_0^{\tilde{s}} (x(\tilde{s}) - x(\tilde{\sigma})) d\tilde{\sigma} .$$  \hspace{1cm} (2)

Here, $b$ is the strip width, $h$ is the thickness of the strip, $x(\tilde{s})$ is the displacement of the strip in horizontal direction, and the mass density of steel is denoted by $\rho$.

2.2. Image processing

To get an accurate measurement, the image distortion has to be corrected and a subpixel interpolation algorithm is used to improve the resolution of the edge detection.

2.2.1. Correction of the image distortion

A simple linear image distortion algorithm is used in this paper. For this purpose, a checkerboard pattern as outlined in Figure 3 is captured by the camera during a calibration procedure. In this checkerboard pattern, each rectangle is partitioned into two triangles.

For the sake of simplicity, just one triangle with the corner-coordinates $(\bar{x}_i, \bar{y}_i)^T$, $i = 1, 2, 3$ is considered. Figure 4 shows such a triangle and its corresponding barycentric coordinates $\beta_1$ and $\beta_2$, see, e.g. [18].

The triangle corner coordinates in world coordinates $(x_w, y_w)$ are known in advance. By capturing an image of the checkerboard pattern and detecting the triangles, the respective triangle corner coordinates can also be easily determined in image pixel coordinates $(x_p, y_p)$. For each triangle, the transformation from barycentric coordinates $(\beta_1, \beta_2)$ to world coordinates can be expressed in the form
Analogously, the image pixel coordinates are given by

\[
\begin{bmatrix}
    x_p \\
    y_p
\end{bmatrix} = A_p \begin{bmatrix}
    \beta_1 \\
    \beta_2
\end{bmatrix} + b_p,
\]

(4)

where \(A_p\) and \(b_p\) are computed in the same way as \(A_w\) and \(b_w\) using the corresponding corner coordinates in the image pixel coordinate system. The inverse transformation to world coordinates then follows as

\[
\begin{bmatrix}
    x_w \\
    y_w
\end{bmatrix} = A_w A_p^{-1} \left( \begin{bmatrix}
    x_p \\
    y_p
\end{bmatrix} - b_p \right) + b_w.
\]

(5)

Before (5) can be evaluated for a certain point \((x_p, y_p)\), the triangle which contains this point has to be found. For this triangle

\[
\beta_1 \geq 0 \land \beta_2 \geq 0 \land (\beta_1 + \beta_2) \leq 1
\]

holds. These inequalities can be used in a naive search for the correct triangle.
2.2.2. **Edge detection**

The multitude of existing edge detecting methods can be classified into subpixel and non-subpixel interpolation algorithms. E.g. the popular Canny edge detector [19] belongs to non-subpixel interpolation methods. It identifies edges at local maxima of the gradient of the image. A major drawback of non-subpixel interpolation methods is that the accuracy is limited to the pixel size of the captured image. In this respect, interpolation-based methods perform better. Here, the position of the edge is estimated within a pixel. Good results can be obtained by the subpixel edge location method based on the partial area effect that was presented in [20]. The corresponding MATLAB source code is freely available\(^1\) and was used to detect the deflection of the strip. For the considered measurement task, detecting the outer strip edge, see Figure 6, proved to be the most reliable way of determining the deflection of the strip. Figure 5 shows a detail of the outer strip edge detected by the Canny and the detected subpixel edge detector. Because its detection results are not discretized, the subpixel edge detector is clearly superior to the Canny edge detector.

2.3. **Approximation of the neutral axis and computation of the bending moment**

Until now, the detection of the outer strip edge was considered. However, the identification of the stress-strain curve requires the coordinates of the neutral axis of the strip. Consider a coordinate system \((\eta, \xi)\), which has its origin at the beginning of the outer strip edge as shown in Figure 6. Furthermore, the coordinate system \((\tilde{x}, \tilde{y})\) is introduced. Again, its origin is at the outer strip edge and the \(\tilde{x}\)-axis points towards the end of the outer strip edge, i.e. the strip-roll contact point. By means of this coordinate system, the detected pixels along the edge are more evenly distributed, which improves the subsequent approximation of the outer strip edge. The detected (and distortion corrected) subpixel coordinates of the outer strip edge are denoted by \((\tilde{x}^i_e, \tilde{y}^i_e)\), \(i = 1, \ldots, n_p\), where \(n_p\) is the number of detected pixels along the edge.

Furthermore, the outer strip edge coordinates \((\tilde{x}^i_e, \tilde{y}^i_e)\) are approximated via piece-wise continuously differentiable polynomials. Based on this approximation, the outer strip edge coordinates can be converted to neutral axis coordinates \((\tilde{x}^i_n, \tilde{y}^i_n)\), see Figure 6. Finally, the neutral axis coordinates \((\tilde{x}^i_n, \tilde{y}^i_n)\) can be easily transformed

![Figure 5. Strip edge coordinates found by the Canny and the subpixel edge detector.](image-url)
back to the \((x, y)\) coordinate system. Let \(y_a(x)\) denote the approximation of the neutral axis of the strip.

In (2), it was shown that the bending moment can be expressed as a function of the normalized strip length coordinate \(\tilde{s}\). To rewrite (2) as a function of \(x\) consider the substitution

\[
\tilde{s} = \sqrt{1 + \left(\frac{dy_a(x)}{dx}\right)^2} \, dx.
\]

Thus, the bending moment \(m_b^i\) at the detected points \(x_n^i\) along the neutral axis reads as

\[
m_b^i(x_n^i) = \frac{m_sg}{b} x_n^i + gph \int_{x_n}^{x_n^i} \sqrt{1 + \left(\frac{dy_a(x)}{dx}\right)^2} \left(x_n^i - \tilde{x}\right) d\tilde{x}.
\]

**Figure 7** illustrates the accuracy of the approximation of the deflection of the strip as well as the bending moment in the strip for an angle of \(\alpha = 90^\circ\).

For this purpose, five piecewise polynomials of degree four proved to be a viable compromise between polynomial degree and approximation accuracy. As can be seen in the bottom part of **Figure 7**, the bending moment caused by the extra load \(m_s\) exceeds the moment caused by the dead load of the strip, cf. (8).

### 3. Identification of the stress-strain curve and Poisson’s ratio

Typically, in tensile test machines, the stress state in the specimen is uniaxial. However, the measurement method proposed in Sec. 1 is characterized by a multiaxial stress state. The Prandtl-Reuß equations are minimized for the conversion between multiaxial stress states and uniaxial stress states, even in case of plastic deformation. Multiaxial stress states do occur in the experiment. Uniaxial stress states are commonly used to describe material parameters. The idea of the proposed identification method is to determine the uniaxial stress-strain curve \(\sigma(\varepsilon)\) and Poisson’s ratio \(\nu\) in such a way that the quadratic error between the measured and the computed deflection of the strip (based on the
identification) is minimized. Therefore, the estimated stress-strain curve to be identified is parametrized by a piecewise linear function with the nodes \((\varepsilon_{e,j}, \sigma_{e,j})\). Here, a monotonous constitutive law \(m_c(\kappa)\) is assumed. Apart from this constraint, the stress-strain curve can have an arbitrary shape.

### 3.1. Prandtl-Reuß equations

In the following, the Prandtl-Reuß equations, cf. [21–23], are used to describe the incremental elasto-plastic deformation of a metal strip. Parts of this section are published in a similar way in [24]. They are repeated here for the sake of completeness. Let \(\sigma_{ij}\) be the components of the stress tensor and \(\delta_{ij}\) the Kronecker delta. Then, the components of the deviatoric stress tensor are

\[
s_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}, \quad i, j, k \in \{1, 2, 3\}.
\]  

The von Mises’ yield condition reads as

\[
s_{ij}s_{ij} - \frac{2}{3} \sigma_Y^2(\varepsilon^p) = 0,
\]  

where \(\sigma_Y(\varepsilon^p)\) is the yield stress and \(\varepsilon^p\) the scalar-valued equivalent plastic strain. In general, \(\sigma_Y(\varepsilon^p)\) is a function of the hardening mechanism, the deformation history, and other influences like the temperature. For the present case, only the hardening mechanism is considered. **Figure 8** shows a typical stress-strain curve obtained by a uniaxial tension test, where a linear elastic material is assumed for \(\varepsilon \leq \varepsilon_Y^0\).
In the plastic region, i.e. $\varepsilon > \varepsilon^0_Y$, the total strain $\varepsilon$ can be divided in the form

$$\varepsilon = \varepsilon^e + \varepsilon^p = \frac{\sigma_Y(\varepsilon^p)}{E} + \varepsilon^p$$  \hspace{1cm} (11)$$

with Young’s modulus $E = \sigma^0_Y/\varepsilon^0_Y$ and $\sigma^0_Y = \sigma_Y(0)$. Using the Prandlt-Reuß equations, the total strain increment $\text{d}\varepsilon_{ij}$ of a material point is defined by

$$\text{d}\varepsilon_{ij} = \frac{1 + \nu}{E} \text{d}\sigma_{ij} - \frac{\nu}{E} \text{d}\sigma_{kk} \delta_{ij} + \frac{3}{2} \frac{s_{ij}}{\sigma_Y(\varepsilon^p)} \text{d}\varepsilon^p.$$  \hspace{1cm} (12)$$

Here, $\nu$ is Poisson’s ratio, and $\text{d}\varepsilon^p$ is the increment of the scalar-valued equivalent plastic strain. The consistency equation relating the stress increments $\text{d}\sigma_{ij}$ and the equivalent plastic strain increment $\text{d}\varepsilon^p$ can be found by calculating the total differential of (10) in the form

$$s_{ij} \text{d}s_{ij} = \frac{2}{3} \sigma_Y(\varepsilon^p) \frac{\text{d}\sigma_Y(\varepsilon^p)}{\text{d}\varepsilon^p} \text{d}\varepsilon^p.$$  \hspace{1cm} (13)$$

In the right hand side of (13), the abbreviation

$$H' = \frac{\text{d}\sigma_Y(\varepsilon^p)}{\text{d}\varepsilon^p}$$  \hspace{1cm} (14)$$

can be introduced. $H'$ is also known as plastic modulus of the material. For the ideal-plastic case, $H' = 0$ holds, whereas $H' > 0$ characterizes the case of a hardening mechanism. Clearly, using (11),

$$\frac{\text{d}\sigma_Y(\varepsilon^p)}{\text{d}\varepsilon^p} = \frac{\text{d}\sigma_Y(\varepsilon^p)}{\text{d}\varepsilon} \left(1 + \frac{1}{E} \frac{\text{d}\sigma_Y(\varepsilon^p)}{\text{d}\varepsilon^p}\right)$$  \hspace{1cm} (15)$$

and (14) reads as

$$H' = \frac{\text{d}\sigma_Y(\varepsilon^p)}{\text{d}\varepsilon^p} = \frac{\frac{\text{d}\sigma_Y(\varepsilon^p)}{\text{d}\varepsilon}}{1 - \frac{1}{E} \frac{\text{d}\sigma_Y(\varepsilon^p)}{\text{d}\varepsilon^p}}.$$  \hspace{1cm} (16)$$

A local coordinate system $x_1$-$x_2$-$x_3$, which is shown in Figure 9, is used for the strip, where the longitudinal direction is represented by $x_1$, the lateral direction by $x_2$ and the transversal direction by $x_3$. In the following, several assumptions are made:
Due to the considered load conditions in the strip, stress components in thickness direction are negligible compared to stress components in the \(x_1-x_2\) plane. The plane state of stress implies \(\sigma_{33} = \sigma_{13} = \sigma_{23} = 0\) and \(d\varepsilon_{13} = d\varepsilon_{23} = 0\) follows from (12). Due to the assumption of zero in-plane shear, shear strains and stresses as well as their corresponding increments vanish, i.e. \(d\varepsilon_{12} = 0\), \(d\sigma_{12} = 0\), \(\varepsilon_{12} = 0\), and \(\sigma_{12} = 0\). Generally, the strip is a shell-like structure with an almost cylindrical shape (\(\kappa < 0\), curvature along the direction \(x_1\)). Because of the curvature-induced spatial dependency of the shell stiffness (bending stiffness along the direction \(x_1 \ll\) bending stiffness along the direction \(x_2\)), it is a very reasonable assumption that the strip curvature along the direction \(x_2\) is negligibly small. By additionally assuming that the mean tensional stress is insignificant compared to bending stresses, \(\varepsilon_{22} = 0\) and \(d\varepsilon_{22} = 0\).

Based on these assumptions and because the strain increment \(d\varepsilon_{33}\) along the thickness direction is not of interest, the remaining considered quantities are the stress increments \(d\sigma_{11}\) and \(d\sigma_{22}\) as well as the equivalent plastic strain increment \(d\varepsilon^p\). In the following, the elastic and the plastic region need to be distinguished. Note that the subsequent relations are only valid for the initial curve of the stress-strain curve. For the elastic region, the inequality

\[
\sigma^0_y > \sqrt{\sigma_{11}^2 - \sigma_{11}\sigma_{22} + \sigma_{22}^2},
\]

\(\text{cf. (10)},\) and \(d\varepsilon^p = 0\) are fulfilled. Hence, it follows from (12) that

\[
d\varepsilon^p = 0
\]

\(\text{(18a)}\)

\[
d\sigma_{11} = \frac{E}{1 - \nu^2} d\varepsilon_{11}
\]

\(\text{(18b)}\)
\[
\frac{d\sigma_{22}}{d\varepsilon_{11}} = \frac{vE}{1 - v^2}.
\] (18c)

If
\[
\sigma_Y^0 \leq \sqrt{\sigma_{11}^2 - \sigma_{11}\sigma_{22} + \sigma_{22}^2} = \sigma_Y(\varepsilon^p),
\] (19)
the deformation is plastic. For this case, combining (12), (13) and (14), the relations
\[
d\varepsilon^p = \frac{2E\sigma_Y}{D} \left( v\sigma_{11} - 2v\sigma_{22} - 2\sigma_{11} + \sigma_{22} \right) d\varepsilon_{11}
\] (20a)
\[
d\sigma_{11} = \frac{4E^2}{D} \left( \sigma_{11}\sigma_{22} - \frac{1}{4}\sigma_{11}^2 - \sigma_{22}^2 - \frac{H^2}{E} \sigma_Y^2(\varepsilon^p) \right) d\varepsilon_{11}
\] (20b)
\[
d\sigma_{22} = \frac{E^2}{D} \left( 5\sigma_{11}\sigma_{22} - 2\sigma_{11}^2 - 2\sigma_{22}^2 - \frac{4H^2}{E} \sigma_Y^2(\varepsilon^p) \right) d\varepsilon_{11}
\] (20c)
with
\[
D = 4H'v^2\sigma_Y^2(\varepsilon^p) + 4Ev\sigma_{11}^2 - 10Ev\sigma_{11}\sigma_{22} + 4Ev\sigma_{22}^2 - 5E\sigma_{11}^2 + 8E\sigma_{11}\sigma_{22} - 5E\sigma_{22}^2 - 4H'\sigma_Y^2(\varepsilon^p)
\] (21)
can be found.

3.2. Optimization

The stress-strain curve to be identified is parametrized by a piecewise linear function with \(n_\sigma\) nodes defined by \((\varepsilon_{e,j}, \sigma_{e,j}), j = 1, \ldots, n_\sigma\). Because the strip curvature is in the range \(\kappa \in \left[-1/(R + h/2), 0\right]\), the relation \(\varepsilon_{11} = -x_3\kappa\) yields
\[
|\varepsilon_{11}| \leq h/(2R + h).
\] (22)

Hence, the attainable strain levels are limited by the radius of the roll with the current strip-roll setup. By reducing the roll radius \(R\), the attainable strain levels can be increased. If even higher strain levels are of interest, the test rig could be adapted in such a way that the roll is eliminated entirely and the strip is clamped at one end (simple cantilever beam). For the identification, the nodes \(\varepsilon_{e,j}\) are equally distributed in the plastic range \((\varepsilon_{e,1} \geq \varepsilon_Y^0\)\), cf. Figure 8) with \(\varepsilon_{e,1} = 0, \varepsilon_{e,2} = \varepsilon_Y^0,\) and \(\varepsilon_{e,n_e} = h/(2R + h)\). The piecewise linear representation of the uniaxial stress-strain curve \(\sigma(\varepsilon)\) is then given by
\[
\sigma(\varepsilon) = \frac{\sigma_{e,j+1} - \sigma_{e,j}}{\varepsilon_{e,j+1} - \varepsilon_{e,j}} (\varepsilon - \varepsilon_{e,j}) + \sigma_{e,j} \quad \text{for} \ \varepsilon \in [\varepsilon_{e,j}, \varepsilon_{e,j+1}].
\] (23)

In the elastic region, \(\sigma_{11} = \sigma(\varepsilon)/(1 - v^2),\) assuming \(\sigma_{11}(0) = 0\). In the plastic region, the nexus between \(\sigma_{11}\) and \(\sigma(\varepsilon)\) is given by (20) considering
\[
\sigma_Y(\varepsilon^p) = \sigma(\varepsilon) = \sigma(\sigma_Y(\varepsilon^p)/E + \varepsilon^p).
\] (24)
To facilitate the inversion of the constitutive law \( m_c(\kappa) \) between the bending moment and the curvature of the strip, a lookup table of the bending moment as a function of the curvature can be calculated by evaluating

\[
m_c(\kappa) = 2 \int_0^\kappa x_3 \sigma_{11} \, dx_3 = 2 \int_0^\kappa \frac{\epsilon_{11}}{\kappa^2} \sigma_{11} \, d\epsilon_{11}.
\]  

(25)

Combining (8) and (25) yields the curvature values

\[
\kappa(x_i^0) = m_c^{-1}(m_b(x_i^0)), \quad i = 1, \ldots, n_p
\]

(26)

at the detected points \( x_i^0 \). The relation between the curvature \( \kappa(x) \) and the bending line \( y_a(x) \) is given by

\[
\kappa(x) = \left( \frac{d^2 y_a}{dx^2} \right)^2 \left( 1 + (d y_a / dx)^2 \right)^{-\frac{3}{2}}.
\]

(27)

This relation can be (numerically) integrated to obtain \( y_a(x) \) after insertion of \( \kappa(x_i^0) \) from (26). In fact, \( \kappa(x) \) denotes an interpolated value from (26) at an arbitrary point \( x \).

With \( \chi_1 = y_a \), (27) can be equivalently expressed as

\[
\frac{d}{dx} \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \begin{bmatrix} \chi_2 \\ \kappa(x) \left( 1 + \chi_2^2 \right)^{\frac{3}{2}} \end{bmatrix}, \quad x \in \left( 0, x_{n p}^0 \right) .
\]

(28)

This differential equation is supplemented by the boundary conditions

\[
\chi_1(x_{n p}^0) = 0 \quad \text{(29a)}
\]

\[
\chi_1(x_{n p}^0) = y_{n p}^0 \quad \text{(29b)}
\]

The solution of (28) and (29) is obtained by the BVP-solver bvp5c [25], which utilizes a multiple shooting method approach. If the identification parameters \( \sigma_{e,j}, j = 1, \ldots, n_\sigma \) and \( \nu \) are summarized in the vector \( p = [\sigma_{e,1}, \ldots, \sigma_{e,n_\sigma}, \nu]^T \), they can be estimated based on the optimization problem

\[
\min_{p \in \mathbb{R}^{n_{var}+1}} \sum_{i=1}^{n_p} \left( y_a(x_{n p}^i) - \chi_1(x_{n p}^i) \right)^2
\]

(30a)

s.t. eqs. (15) – (21), (23) – (29)

\[
\sigma_{e,1} = 0
\]

(30c)

\[
\frac{dm_c(\kappa)}{d\kappa} \leq 0.
\]

(30d)

The constraints (30c) and (30d) ensure a stress-strain curve \( \sigma(e) \) that begins at \( \sigma(0) = 0 \) and a monotonous constitutive law \( m_c(\kappa) \). Because of \( \kappa < 0 \) and \( m_b > 0 \), \( m_c(\kappa) \) needs to be monotonously decreasing. The optimization problem (30) is solved by an interior point method in MATLAB with the command fmincon.
4. Verification of the identification method

Laboratory experiments were carried out to demonstrate the feasibility of the proposed measurement and identification method. In this section, the results of the experiments are shown for a typical material sample. Results with similar accuracy have been obtained for other materials. The parameters of the test specimen, which is a cold rolled high strength low alloyed fine-grained steel with $\leq 0.1\%C$, and the test rig are listed in Table 1. To verify the obtained results, a second specimen, taken from the same steel sheet, was measured using the tensile testing machine 'Beta 100' of Messphysik materials testing GmbH and compared to the measurements of the proposed test rig. A picture of the tensile testing machine is shown in Figure 10. The machine uses a laser speckle array extensometer to measure the strain in the specimen. Furthermore, the specifications of the tensile testing machine and the dimensions of the specimen are given in Table 2.

Figure 11 shows the stress-strain curves obtained by the tensile test machine and the proposed identification method, where the stress-strain curve was estimated based on a single measurement with $\alpha = 90^\circ$. The comparison shows a good agreement throughout the entire measuring range for both methods. Moreover, Poisson’s ratio was estimated to be $\nu = 0.3177$.

Remark 1 (Verification of the bending line) A further verification possibility is to compare the computed and the measured bending line of the strip for other angles than that used in the identification procedure. Therefore, a boundary value problem has to be derived and the constitutive law $m_c(\kappa)$ must be computed based on the identified stress-strain curve $\sigma(\varepsilon)$ by evaluating (25). Without showing the results explicitly, a good agreement between measurement and simulation is observed for the scenario considered in this paper.

5. Conclusions

A simple method to identify the stress-strain curve of a metal strip was proposed and verified by a standard tensile test. The main advantages of the proposed method are the simple set-up and its low costs. Despite the simplicity and low effort of the method, its accuracy was found to be absolutely sufficient for standard material testing applications. In summary, the following pros (+) and cons (−) can be attributed to the proposed method:

+ inexpensive, quick setup
+ robust (only integration is necessary, no derivatives)
+ problems of tensile test machines (strain sensor, etc.) are avoided
+ wide range of plasticity models is possible
− calculation effort

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass density of steel</td>
<td>$\rho$</td>
<td>7850 kg/m$^3$</td>
</tr>
<tr>
<td>Gravitational acceleration</td>
<td>$g$</td>
<td>9.81 m/s$^2$</td>
</tr>
<tr>
<td>Radius of roll</td>
<td>$R$</td>
<td>150 mm</td>
</tr>
<tr>
<td>Mass of extra load</td>
<td>$m_s$</td>
<td>3.2 kg</td>
</tr>
<tr>
<td>Length of strip</td>
<td>$L$</td>
<td>751 mm</td>
</tr>
<tr>
<td>Width of strip</td>
<td>$b$</td>
<td>130 mm</td>
</tr>
<tr>
<td>Thickness of strip</td>
<td>$h$</td>
<td>1.253 mm</td>
</tr>
</tbody>
</table>
Figure 10. Detail of the tensile testing machine ‘Beta 100’ of Messphysik materials testing GmbH (without specimen).

Table 2. Specifications of the tensile testing machine ‘Beta 100’ and corresponding dimensions of the specimen.

<table>
<thead>
<tr>
<th>Quantity</th>
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<tbody>
<tr>
<td>Length of specimen</td>
<td>0.3 mm to 3 mm</td>
</tr>
<tr>
<td>Width of specimen</td>
<td>20 mm</td>
</tr>
<tr>
<td>Effective clamping width</td>
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<tr>
<td>Effective specimen length</td>
<td>120 mm</td>
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<tr>
<td>Load capacity</td>
<td>100 kN</td>
</tr>
<tr>
<td>Resolution of position-control</td>
<td>0.007 µm</td>
</tr>
<tr>
<td>Resolution with full capacity load cell</td>
<td>0.83 N</td>
</tr>
</tbody>
</table>
conversion from uniaxial stress-strain curve to general stress state necessary
attainable strain levels are limited by the radius \( R \) of the roll.

**Note**


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**Disclosure statement**

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**References**


![Figure 11. Comparison of the tensile test and the identified stress-strain curve.](image-url)


Appendix. Computation of the stress-strain curve

This section is concerned with the computation of the stress-strain curve assuming that the curvature-moment relation is known. The constitutive law (25) can be expressed in the form

$$\kappa^2 m_c(\kappa) = 2 \int_{-\frac{h}{2}}^{\frac{h}{2}} \varepsilon_{11} \sigma_{11} (\varepsilon_{11}) d\varepsilon_{11}. \quad (A1)$$

Differentiating (A1) with respect to $\kappa$ yields

$$2\kappa m_c(\kappa) + \kappa^2 \frac{dm_c(\kappa)}{d\kappa} = \frac{kh^2}{2} \sigma_{11} \left( -\frac{kh}{2} \right). \quad (A2)$$

Consequently, the stress-strain curve is given by

$$\sigma_{11} \left( -\frac{kh}{2} \right) = \frac{2}{h^2} \left( 2m_c(\kappa) + \kappa \frac{dm_c(\kappa)}{d\kappa} \right). \quad (A3)$$