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ABSTRACT
In portfolio optimization a classical problem is to trade with assets so as to maximize some kind of utility of the investor. In our paper this problem is investigated for assets whose prices depend on their past values in a non-Markovian way. Such models incorporate several features of real price processes better than Markov processes do. Our utility function is the widespread logarithmic utility, the formulation of the model is discrete in time. Despite the problem being a well-known one, there are few results where memory is treated systematically in a parametric model. Our algorithm is optimal and this optimality is guaranteed for a rich class of model specifications. Moreover, the algorithm runs online, i.e., the optimal investment is achieved in a day-by-day manner, using simple numerical integration, without Monte-Carlo simulations. Theoretical results are demonstrated by numerical experiments as well.

1. Introduction
Modern portfolio optimization started with Markowitz’s theory Markowitz (1952), barely 20 years after Kolmogorov had laid the foundations of probability theory. In 1956, J. L. Kelly (1956) formulated a strong connection between information rate and portfolio theory – this is the subject of the present paper as well.

A portfolio is a collection of financial assets (e.g., stocks, bonds) held by an investor whose intention is to optimize some kind of benefit from this ownership. For long-term investors, the most important property of a portfolio is its return distribution in the future, but it is not evident how this must be taken into consideration. To maximize its expected value seems an obvious idea, but this approach avoids addressing risk and may actually lead to disaster in the long run, see e.g., Chapter 16 of Cover and Thomas (2006).

A better approach is to maximize the expected logarithm (called utility function) of the portfolio value. This idea originated from Daniel Bernoulli, see Bernoulli (1954). While plain expectation is a generalization of the arithmetic mean, the logarithm transforms it into a geometric mean which is more sensitive to the fluctuations in the future value, i.e., to risk. Log-optimal portfolio beats every other portfolio in the long run, see e.g., Chapter 16 of Cover and Thomas (2006).

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Famous investors who used log-optimal investments include Jim Simons and Warren Buffett from Renaissance Medallion as well as the economist John Maynard Keynes.

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The evolution of stock prices in time can be regarded as a (stochastic) dynamical system where the state variables are described by e.g., a (stochastic) differential equation. Such models implicitly assume that it is enough to know the structure of the dynamics and the actual state to make calculations for the future (i.e., the Markovian property is assumed). The theory of portfolio management is well-studied in this case, where the current state (or the past few states) contains all the information we can have about the distribution of future prices. On the other hand, why shouldn’t we take into account all the information from past data, as empirical evidence suggests (see e.g., Comte and Renault (1998); Gatheral, Jaisson, and Rosenbaum (2018); Ding, Granger, and Engle (1993); Andersen and Bollerslev (1997); Andersen et al. (2001))?

For theoretical investigations it is common to use continuous-time models but they have to be turned into discrete time to make numerical investigations possible. As there is no hope for closed-form solutions in complex, non-Markovian systems, the only possibility is computations. For this reason we use discrete time financial models in this paper, and within this setup we propose an online optimal growth rate investment method for certain non-Markovian market models.

In what follows we will work with the log-return process, i.e., the increments of the log-price and consider various specifications for its dynamics that are not Markovian. For an overview about the statistical properties of prices, see Cont (2001). The most important property for us is the structure of time correlation which was studied already in the 1990s, see e.g., Ding, Granger, and Engle (1993); Andersen and Bollerslev (1997); Andersen et al. (2001).

In the last 50 years the theory of log-optimal investments has been developed in a wide range of models, but with very little practical implication for the non-Markovian case. The handbook by MacLean, Thorp, and Ziemba (2011) collected many important papers concerning the topic. We also mention the milestone paper Algoet and Cover (1988), who characterized the log-optimal portfolio for stationary and ergodic log-return processes. However, this is a purely theoretical result that cannot be implemented in general.

Practical results for log-optimal investments in a distribution-free setting were investigated by Györfi et al., where the optimal strategy is achieved by machine learning methods, for a survey, see Györfi, Ottucsák, and Urbán. (2012). Our approach, in contrast to the distribution-free solution, uses parametric models, where general properties of stock prices can be taken into consideration. Also, with parameters we are able to characterize the memory effect in the models in a simple way. We are also able to characterize how the optimal strategy or the value function depends on the memory.

The structure of the paper is the following. In Section 2 we setup the investment problem, then we define the concept of log-optimal portfolio and prove a theorem about the characterization of the optimal strategy. In Section 3 we introduce two approximative strategies. In Section 4 we provide two stock price models with different statistical properties but both of them carry a significant memory effect and can be analysed using our method. In Section 5 we prove limit theorems for our models in the long-run investment problem and thus make the investment strategy computationally feasible. The last section presents numerical results, where we interpret the optimal decision the investor should make.
2. Log-Optimal Portfolios

This section contains the financial setting and a theorem about log-optimal investments in both the finite time and the long-run case. We use discrete time models and the time parameter will be $t \in \mathbb{Z}$ or $t \in \mathbb{N}$.

For simplicity, we deal only with one risky and one riskless asset. We denote the stock (risky asset) price by $S_t$ and the bond (riskless asset) price by $B_t$ at time $t$, with initial prices $s_0$ and $b_0$, respectively.

Investment starts at $t = 0$ and the bond price evolves in time deterministically as $B_t = b_0(1 + r)^t$, $t \in \mathbb{N}$, where $r \geq 0$ is the fixed interest rate. It is convenient to describe the stock dynamics by its log-return $H_t := \log(S_t/S_{t-1})$ whose statistical properties are well-documented, see e.g., Cont (2001).

The process $S_t$ is defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{Z}}, \mathbb{P})$ where the $\sigma$-algebra $\mathcal{F}_t$ is generated by $\{\epsilon_j\}_{j \in \mathbb{Z}}$ and $\{\eta_j\}_{j \in \mathbb{N}}$, where $\epsilon_j$, $j \in \mathbb{Z}$ are standard Gaussian. As it will become clear below, we will treat processes that behave in a Markovian way conditional to a Gaussian process. In other words, we treat certain Markov chains in Gaussian random environments where the latter will be generated by the sequence $\epsilon_j$, $j \in \mathbb{Z}$.

The investor’s strategy will be described by a $[0, 1]$-valued adapted process $\pi_t$, $t \in \mathbb{N}$ which represents the proportion of wealth allocated to the stock. More precisely, for initial wealth $W_0 = w_0$ is a constant and $w_0 > 0$ and for any strategy $\pi_t$, the investor’s wealth $W_t^\pi$ at time $t$ follows the evolution

$$W_{t}^{\pi} = W_{t-1}^{\pi} \left( (1 - \pi_{t-1}) \frac{B_t}{B_{t-1}} + \pi_{t-1} \frac{S_t}{S_{t-1}} \right).$$

To specify the price evolution of the stock, we assume that the log-return process is of the form

$$H_t = F(Z_{t-1}, Y_t, \epsilon_t, \eta_t),$$

where $F$ is a measurable function, $Y_t$ is a stationary Gaussian process such that $Y_t$ is $\sigma(\epsilon_j, j \leq t)$-measurable. Here $Z_{t-1}$ contains the information about the past values of the log-return, so it is assumed that $Z_{t-1} = j(H_{t-1}, H_{t-2}, \ldots)$ for some measurable $j : \mathbb{R}^\mathbb{N} \to \mathbb{B}$ for some Banach space $\mathbb{B}$. For instance, in the simplest case, we have $Z_{t-1} = H_{t-1}$ and $\mathbb{B} = \mathbb{R}$. The process $Y_t$ represents some economic factor, e.g., the log-volatility. For simplicity, we assume from now on that $Y_t$ is a causal linear process, i.e.,

$$Y_t = \sum_{j=0}^{\infty} a_j \epsilon_{t-j},$$

with some coefficients $a_j$ satisfying $\sum_{j=0}^{\infty} a_j^2 < \infty$. We also assume $a_0 \neq 0$.

Notice that the random process $Y_t$, $t \in \mathbb{N}$ is also conditionally Gaussian, since the conditional distribution of $Y_t$ on the $\sigma$-algebra $\mathcal{F}_{t-1}$ is Gaussian. The key observation is
that its conditional law is then determined by the conditional expectation and the conditional variance processes:

\[ v_{t-1} := \mathbb{E}[Y_t | \mathcal{F}_{t-1}] \quad \text{and} \quad \kappa^2_{t-1} := \text{Var}(Y_t | \mathcal{F}_{t-1}), \quad t \in \mathbb{Z}. \quad (3) \]

Remark 1. It will become clear that for our arguments below it suffices to assume that \( Y_t \) is a conditionally Gaussian process. However, we stay with the simple linear case for the sake of a transparent presentation. Clearly, for the case of a linear process \( \kappa_t = \kappa = a_0^2 \) constant. We nevertheless keep the notation \( \kappa_t \) for a while to indicate that more general processes could be similarly treated.

As we can see, both variables \( (Z_{t-1} \text{ and } Y_t) \) can bring about a memory effect (i.e., they can have memory, unlike a Markovian process).

In Section 4 we are going to show two examples within the model class presented here and describe their connection with well-known models and practical usage.

For us, the main property of the models considered here is that they allow a non-Markovian structure, indicated many times by statistical investigations. Such a model is general enough to describe important phenomena, such as leverage effect, correlation between volatility and driving noise, kurtosis, skewness, anti-persistence, etc. Sampling from certain famous continuous-time models (see e.g., Gatheral, Jaisson, and Rosenbaum (2018)) also leads to discrete-time models of the type treated here.

The objective of investment is to maximize the utility function by choosing the strategy \( \pi_t \) adequately. Typically two cases can be optimized in the log-optimal portfolio problem: (i) finite horizon problem, where we want to optimize the utility function at a fixed time \( T > 0 \); or (ii) the long-run problem, where we optimize the time-average of the log-wealth. The optimal strategy \( \pi^*(T) := \{\pi^*_t(T)\}_{t=0}^{T-1} \) for the finite horizon problem (if exists) satisfies

\[ \mathbb{E}\left[ \log \left( W_T^{\pi^*(T)} \right) \right] = \sup_{\pi} \mathbb{E}\left[ \log(W_T^{\pi}) \right], \quad (4) \]

where the supremum is over \( [0,1] \)-valued adapted processes \( \pi = (\pi_0, \ldots, \pi_{T-1}) \). The long-run problem is defined similarly, the optimal strategy \( \pi^* := \{\pi^*_t\}_{t=0}^{\infty} \) is such that

\[ \lim_{T \to \infty} \inf_{\pi} \frac{1}{T} \mathbb{E}\left[ \log \left( W_T^{\pi^*} \right) \right] = \max \lim_{T \to \infty} \inf_{\pi} \frac{1}{T} \mathbb{E}\left[ \log(W_T^{\pi}) \right], \quad (5) \]

where \( \pi_t, \ t \in \mathbb{N} \) ranges over adapted \( [0,1] \)-valued processes. The characterization of optimal strategy will be obtained thanks to the conditionally Gaussian dynamics we assumed about \( Y_t \). We will furthermore use ergodic arguments for the long-run case.

The log-optimal portfolio is also called growth-optimal portfolio because it is attained by optimizing the expected logarithm of the growth function at each investment decision. We introduce the expected growth function \( g^\pi \):

\[ g^\pi(z, \nu, \kappa) := \mathbb{E} \left[ \log \left( (1 - \nu)(1 + r) + \pi e^{F(z, \nu + \kappa \xi_0, \eta_0, \xi_1, \eta_1)} \right) \right] \quad (6) \]

Here \( \nu \) and \( \kappa \) are real numbers, \( \pi \in [0,1] \), \( z \in \mathbb{R} \); the random variables \( (\varepsilon_0, \eta_0) \) have the same distribution as \( (\varepsilon_t, \eta_t), \ t \geq 1 \).
Lemma 2.1. There exists a constant $c_1 > 0$ such that, for all $\pi \in [0, 1]$,

$$\left| \log \left( (1 - \pi)(1 + r) + \pi e^{F(z, \nu + \kappa \epsilon_0, \eta_0)} \right) \right| \leq c_1 (1 + |F(z, \nu + \kappa \epsilon_0, \epsilon_0, \eta_0)|).$$

Proof. Clearly,

$$(1 - \pi)(1 + r) + \pi e^{F(z, \nu + \kappa \epsilon_0, \eta_0)} \leq 1 + r + e^{F(z, \nu + \kappa \epsilon_0, \eta_0)} \leq 2e^{\max \{F(z, \nu + \kappa \epsilon_0, \epsilon_0, \eta_0) \log(1+r)\}} \leq 2e^{F(z, \nu + \kappa \epsilon_0, \epsilon_0, \eta_0) + \log(1+r)}.$$ 

If $\pi > 1/2$ then

$$(1 - \pi)(1 + r) + \pi e^{F(z, \nu + \kappa \epsilon_0, \eta_0)} \geq e^{F(z, \nu + \kappa \epsilon_0, \eta_0)} / 2.$$

If $\pi \leq 1/2$ then

$$(1 - \pi)(1 + r) + \pi e^{F(z, \nu + \kappa \epsilon_0, \eta_0)} \geq (1 + r) / 2.$$

The statement follows easily from these estimates. □

From now on we concentrate on the case when $Z_{t-1} = H_{t-1} \in \mathbb{B} := \mathbb{R}$, but the following lemmas in the finite dimensional case $Z \in \mathbb{R}^k$ would be similar. We will briefly comment on the general case later. □

Assumption 2.2. Assume that $F$ is continuous, $Z \in \mathbb{R}$ and for each $N > 0$,

$$\sup_{z, \nu, \kappa \in [-N, N]^3} |F(z, \nu + \kappa \epsilon_0, \epsilon_0, \eta_0)| \leq X(N)$$

with some integrable random variable $X(N)$.

Lemma 2.3. Let Assumption 2.2 be in force. Then the function

$$(z, \nu, \kappa, \pi) \rightarrow g^\pi(z, \nu, \kappa)$$

is continuous.

Proof. It is enough to prove continuity on every cube $[-N, N]^3$, $N > 0$. Now dominated convergence and Lemma 2.1 show the statement. □

Lemma 2.4. Under Assumption 2.2, there exists a measurable function $(z, \nu, \kappa) \rightarrow \hat{\pi}(z, \nu, \kappa) \in [0, 1]$ such that

2Strictly speaking, $\kappa$ should range over $[0, \infty)$. However, the law of $\epsilon_0$ being symmetric, one may equally well let $\kappa$ run over $\mathbb{R}$.
\[ g_{\bar{\pi}(z,v,\kappa)}(z, v, \kappa) = \sup_{\pi \in [0,1]} g^\pi(z, v, \kappa). \]

Proof. We will use Proposition D.5 of Hernández-Lerma and Lasserre (1996) with the choice \( \psi(z, v, \kappa) := [0, 1] \) and \( v(z, v, \kappa, \pi) := g^\pi(z, v, \kappa) \). Note that \( \psi \) is trivially a compact-valued Borel-measurable multifunction and that \( v \) is continuous by Lemma 2.1 so the cited proposition indeed applies and we can conclude. \qed

The theorem below gives the optimal solution for our investment problem, which relies on a day-by-day (on-line) optimization procedure where the investor uses the information in \( F_{t-1} \) to achieve optimality at time \( t \). \qed

Theorem 2.5. Let us regard a log-optimal portfolio problem defined in (4) with the dynamics given by (2). The following strategy is log-optimal, on finite as well as on infinite horizons:

\[ \pi^*_t = \hat{\pi}(Z_{t-1}, v_{t-1}, \kappa_{t-1}), \quad t = 1, 2, \ldots, T. \]

Remark 2. In order to use the above theorem, the investor needs the quantities \((Z_{t-1}, v_{t-1}, \kappa_{t-1})\) but these are available information at \( t-1 \). For calculating \( \hat{\pi} \), the investor has to evaluate the function \( g^\pi \) at every value of \( \pi \in [0,1] \) but in practice it is enough to do this along a finite partition of the interval. Calculation of \( g^\pi \) is easy, since it is the expected value of a function of a two-dimensional random variable \((\varepsilon_0, \eta_0)\).

Proof. We need to prove that \( \mathbb{E}[\log(W^\pi_T)] = \sup_{\pi} \mathbb{E}[\log(W^\pi_T)] \) for the strategy \( \pi^* \) defined in the statement of Theorem 2.5. W. l. o. g. let \( w_0 := 1 \). Using the identity

\[ W^\pi_T = \prod_{t=1}^T \frac{W^\pi_t}{W^\pi_{t-1}} = \prod_{t=1}^T \exp(G_t), \]

we get that

\[ \mathbb{E}[\log W^\pi_T] = \sum_{t=1}^T \mathbb{E}[G^\pi_t] = \sum_{t=1}^T \mathbb{E}[\mathbb{E}[G^\pi_t | F_{t-1}]]. \]

Since the sum is finite it is enough to maximize each term in the sum individually, i.e.,

\[ \mathbb{E}[G^\pi_t | F_{t-1}] \rightarrow \max. \]

Notice the identity

\[ \text{Law}[G^\pi_t | F_{t-1}] = \text{Law}\left[\log\left((1 - \pi)(1 + r) + \pi \varepsilon^F(z,v+\kappa \varepsilon_0,\varepsilon_0,\eta_0)\right)\right]_{z=Z_{t-1},v=v_{t-1},\kappa=\kappa_{t-1}}. \]

Hence it is enough to maximize the expected value of the right-hand side above which is exactly what \( \hat{\pi}(Z_{t-1}, v_{t-1}, \kappa_{t-1}) \) does. \qed

The following assumption will be important to numerically implement the long-run optimization procedure. \qed
Assumption 2.6. Let Assumption 2.2 be valid. Denote

\[ Q_t := g^{\tilde{\pi}(Z_t, \nu_t, \kappa_t)}(Z_t, \nu_t, \kappa_t), \quad t \in \mathbb{N} \]

where \( \tilde{\pi} \) is as in Lemma 2.4. The limit \( \lim_{N \to \infty} \left( \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{E}Q_t \right) / N \) exists, the limit \( \lim_{N \to \infty} \left( \sum_{i=0}^{N-1} Q_t \right) / N \) also exists almost surely and the two limits are equal.

When we mention ‘ergodicity’ in the sequel, what we really mean is the fulfilment of Assumption 2.6 in the given model.

Remark 3. In the literature the above mentioned limit is called growth-rate \( G^* \), that is

\[ G^* := \lim_{N \to \infty} \left( \frac{1}{N} \sum_{i=0}^{N-1} \mathbb{E}Q_t \right) / N. \] (8)

Of course, in a case when \( H_t \) is stationary, we have \( G^* = \mathbb{E}Q_t \).

Remark 4. Similar results hold when \( \mathbb{B} \) is a Banach space of sequences, e.g., \( l^2 \). This is the case for the bilinear ARCH model of Subsection 4.2 below. For simplicity, we do not treat the corresponding theory here.

3. Approximate Log-Optimal Portfolios

The log-optimal solution is achieved by optimizing the expected value in (6). To calculate the expected value with respect to \( \varepsilon_0 \) and \( \eta_0 \), we need to numerically evaluate a two-dimensional integral for a given \( \pi \). When there are \( N \) stocks – a case which we will not touch on in this paper – the numerical integral becomes \( 2N \)-dimensional. To overcome this difficulty it is desirable to approximate the log-optimal strategy with a sub-optimal one which is computationally more feasible. We will see that these approximations also help in understanding the log-optimal strategy. For simplicity, we choose the interest rate \( r = 0 \).

In the literature a second order approximation has been defined in Vajda (2006) under the name semi-log-optimal strategy. In Li and Hoi (2014) and Győrfi, Urbán, and Vajda (2007) the method was investigated numerically in a non-parametric model setting. We do not use this terminology since we will show another kind of approximation (first-order) as well, and it might be confusing as to what the semi-log-optimal portfolio refers to.

We define the linearly approximated log-optimal portfolio when we approximate the function \( \log(\cdot) \) and \( \exp(\cdot) \) in (6) by their first order Taylor polynomial, and quadratic approximated log-optimal portfolio when we approximate by their second order Taylor polynomial. Higher-order terms are neglected. We denote these strategies as \( \tilde{\pi}^{lin} \) and \( \tilde{\pi}^{quad} \). The approximated growth functions in a general stock price model with log-return \( H_t \) have the following form:
These expressions are polynomial functions of $\pi$, therefore their optima can be calculated easily. In the linear case the approximative optimal strategy is:

$$\pi_{\text{lin}}^{\pi} := \begin{cases} 1, & \text{if } \mathbb{E}[H_t | \mathcal{F}_{t-1}] > 0 \\ 0, & \text{otherwise}. \end{cases}$$

In the quadratic case the approximative optimal strategy is:

$$\pi_{\text{quad}}^{\pi} = \frac{2\mathbb{E}[H_t | \mathcal{F}_{t-1}]}{\mathbb{E}[H_t^2 | \mathcal{F}_{t-1}]} - 1$$

restricted on the $[0, 1]$ interval.

Of course, these conditional expectation formulas are not much easier to calculate than the original formula in (6). In the Conditionally Gaussian model class however, where $H_t = F(H_{t-1}, Y_t, \epsilon_t, \eta_t)$, the conditional expectation can be reduced in the same way to an unconditional expectation. We have

$$\mathbb{E}[H_t | \mathcal{F}_{t-1}] = \mathbb{E}[F(z, \nu + \kappa \epsilon_0, \epsilon_0, \eta_0)]_{z \equiv H_{t-1}, \nu \equiv Y_{t-1}, \kappa \equiv \kappa_{t-1}},$$

which is, again, a two-dimensional integral in $\epsilon_0$ and $\eta_0$.

In specified parametric models, these formulas can become simpler. In the following sections we will study two models where these approximative strategies can be calculated as a simple function evaluation.

Remark 5. In the linearly approximated log-optimal portfolio the optimal strategy is extreme, i.e., its value is 0 or 1, but there is no mixed strategy. The strategy in the Conditionally Gaussian model class is a function of $h$, $\nu$ and $\kappa$: $\tilde{\pi}_{t}^{\text{lin}} = \tilde{\pi}_{t}^{\text{lin}}(h, \nu, \kappa)$. For several models, as we will see later, we can define a $0\text{-}1$ barrier function denoted as $\tilde{h}(\nu, \kappa)$ and approximate $\pi_t^*$ by

$$\tilde{\pi}_{t}^{\text{lin}} = \begin{cases} 1, & \text{if } H_{t-1} > \tilde{h}(\nu_{t-1}, \kappa_{t-1}) \\ 0, & \text{otherwise}. \end{cases}$$

Remark 6. There is an interesting parallel between the strategy of the second order approximation and the log-optimal solution when stock price is a diffusion in continuous time. If the stock price in the diffusion case is described by $dS_t/S_t = \mu(S_t)dt + \sigma(S_t)dW_t$, $S_0 = s_0$, for some real-valued functions $\mu(\cdot)$ and $\sigma(\cdot)$ and a Brownian motion $W_t$, then the optimal solution is $\pi_t^{*, \text{diffusion}} = \mu(S_t)/\sigma(S_t)$ (here, it is not restricted between 0 and 1). See the optimal strategy in Subsection 1.2 at page 6 in Davis and Lleo (2014). It is very similar to the quadratic approximative solution in (10).
4. Stock Price Models

In this section we give two examples for stock price dynamics in order to illustrate the optimization method described in the previous section. In the literature, price models have been extensively studied, for example in Shephard (2005) and Andersen et al. (2009). There are two main types of models according to the handbook of Andersen et al. (2009), Stochastic Volatility models and GARCH-like models and we will present one of each type. Both types have a tremendous amount of examples already investigated but here we focus on those which capture the (long) memory effect in a simple and effective way.

In GARCH-like models the choice is clear: long memory is typically described by the Linear ARCH model (LARCH) or its extension, the Bilinear ARCH model. For this reason we use the expression ‘GARCH-like’, since LARCH and BARCH are not part of the GARCH models, but are typically mentioned as a variant of it. Therefore one of the two examples we give is the BARCH model. In Stochastic Volatility models long memory is described both in continuous time and discrete time. Two widely used examples are the Fractional Stochastic Volatility (FSV) model (in continuous time) by Comte and Renault (1998) and Long-Memory Stochastic Volatility (LMSV) model by Breidt, Crato, and De Lima (1998) (in discrete time, see also Comte, Lacour, and Rozenholc (2010)). Our model proposition in the Stochastic Volatility framework can be regarded as a discretized version of the FSV as well as a more general formalization of LMSV; we call it Discrete Gaussian Stochastic Volatility (DGSV).

We not only define these two models in this section, but we also use heuristically chosen parameters based on real data and characterize the memory properties of the models. Clearly, our choice of models is somewhat arbitrary: yet it serves to demonstrate that even rather complex models easily fit into the model class we presented in Section 2.

A drawback of GARCH type models is that they typically require many parameters which are difficult to estimate; it is not easy to assign a financial meaning to them.

Continuous-time models, on the other hand, usually use fractional Brownian motion which, in contrast, captures the memory effect with only one parameter. The related portfolio optimization problems are, however, not solved (except some very particular examples like in Guasoni, Nika, and Miklos (2017)).

Our model settings enable us to include memory in a simple manner in such a way that the memory can be characterized rather well by one parameter. At the same time, financial meaning can be assigned to the other parameters as well.

Before we continue with the definitions of the models, we define three types of memory here. Let us call the auto-correlation function \( y_X(k) \) of a stationary stochastic process \( \{X_t\}_{t \in \mathbb{Z}} \) with finite second moments:

\[
y_X(k) := \frac{\mathbb{E}[(X_t - \mathbb{E}[X_t])(X_{t+k} - \mathbb{E}[X_{t+k}])]}{\mathbb{E}[(X_t - \mathbb{E}[X_t])^2]}.
\]

We define the types of memory based on the tail-behaviour of the auto-correlation function. There is no universally agreed definition for long memory.

Definition 4.1. A random process \( X_t \) with auto-correlation function \( y_X(k) \) has
(1) long memory, if $\sum_{k=1}^{\infty} y_X(k) = \infty$;

(2) moderate memory, if $\sum_{k=1}^{\infty} |y_X(k)| < \infty$ and $\lim_{k \to \infty} y_X(k)k^l = c(k) > 0$ for some $l > 1$;

(3) short memory, if $\sum_{k=1}^{\infty} |y_X(k)| < \infty$ and $\lim_{k \to \infty} y_X(k)k^l = 0$ for every $l > 1$.

Processes with moderate memory are important because their autocorrelation decays as a power law function and not exponentially, therefore its effect is stronger. In several time-series models ergodicity can only be proven for moderate memory processes but not for long memory processes (see for example Giraitis and Surgailis (2002)).

Here, we will precisely define our two examples by giving exact formulas for (2). We remind the reader that $Z_{t-1}$ is a function of past log-return data $\{H_{t-1}, H_{t-2}, \ldots\}$ and $Y_t$ is a conditionally Gaussian process with driving noise $\{\varepsilon_t, \varepsilon_{t-1}, \ldots\}$. From now on we assume that $Z_{t-1}$ is a (possibly infinite) linear combination of the $\{H_{t-1}, H_{t-2}, \ldots\}$ and $Y_t$ is a linear process (hence $\kappa_t = \kappa$, a constant).

4.1. Discrete Gaussian Stochastic Volatility

The first model is the DGSV which was conceived to have similar behaviour to the continuous-time FSV model (see Comte and Renault (1998) and Gatheral, Jaisson, and Rosenbaum (2018)). The dynamics of the model is made up of two parts: the drift, which is a linear function of the past log-return $H_{t-1}$ and the volatility, which depends on $Y_t$ and may have long memory. The process $Y_t$ here is called log-volatility and it is an arbitrary casual linear process driven by $(\varepsilon_j, j \leq t)$. The dynamics of the log-return is:

\[
H_t = \mu + \alpha H_{t-1} + \sigma \varepsilon_t \left( \rho \varepsilon_t + \sqrt{1 - \rho^2} \eta_t \right) , \quad \mu \in \mathbb{R}; \tag{13a}
\]

\[
Y_t = \sum_{j=0}^{\infty} \beta_j \varepsilon_{t-j}, \quad \beta_j \in \mathbb{R}. \tag{13b}
\]

The constant $\rho \in [-1, 1]$ is the coefficient of the leverage effect, i.e., the correlation between the driving noises of the log-price and the volatility. We assume the mean-reverting coefficient $|\alpha| < 1$ and the volatility coefficient $\sigma > 0$. The noise sequence $(\varepsilon_j, j \leq t)$ must be Gaussian in the Conditional Gaussian class, although the sequence $(\eta_j, j \leq t)$ is not necessarily Gaussian, but its distribution must be known in order to calculate $\pi$ in Section 2.

This is reasonable if we wish $H_t$ to be ergodic. The general form of the log-volatility describes a rather wide range of possible processes. In the following sections we restrict its parameters to $\beta_j = b_0(1 + j)^{-b}$, $j \in \mathbb{N}$. This simple form allow us to characterize the decay of the memory by one parameter, $b$. Also, $b_0$ plays an important role: we will see that it not only effects the strength of the memory but also the conditional skewness and kurtosis. Here we will mainly investigate the case with long memory where $b$ is between $(0.5, 1)$. If $b > 1$, it corresponds to the moderate memory case.
We defined the log-return as $H_t = F(H_{t-1}, Y_t, \epsilon_t, \eta_t)$ with $F(h, y, \epsilon, \eta) = \mu + \alpha h + \sigma \exp(y)(\rho \epsilon + \sqrt{1 - \rho^2} \eta)$. The conditional mean and variance of the log-volatility $Y_t$ are

\begin{equation}
\nu_{t-1} = \mathbb{E}[Y_t|\mathcal{F}_{t-1}] = b_0 \sum_{j=1}^{\infty} (1 + j)^{-b} \mathbb{E}_t^{-j}, \quad (14a)
\end{equation}

\begin{equation}
\kappa_{t-1} = \text{Var}(Y_t|\mathcal{F}_{t-1}) = b_0. \quad (14b)
\end{equation}

Calculating these at every $t$, we can determine the log-optimal strategy by Theorem 2.5. We emphasize that this model can both have long memory and can be ergodic at the same time.

Besides the other advantageous properties that we formerly mentioned, this model is also able to capture basic moment-related properties. In most stock price models with long-memory effect it is usually impossible to calculate the (conditional/unconditional) moments, while in our model not only can it be done, but in addition, moments can be calculated explicitly.

To calculate the $k$-th moment it is enough to know $\mathbb{E}[e^{Kt}\epsilon]$ for every positive integer $j \leq k, K \in \mathbb{R}$, where $\epsilon \sim \mathcal{N}(0, 1)$. By a simple substitution of the variable in the integral of the expression of expected value, we get that $\mathbb{E}[e^{Kt}\epsilon] = e^{-K^2/2} \mathbb{E}[(\epsilon + K)^j]$, which can be calculated explicitly. E.g., for $j = 1$ it is equal to $e^{-K^2/2}K$, for $j = 2$ it is equal to $e^{-K^2/2}(1 + K^2)$. The calculations are elementary but long; we do not detail them here, but we do show the most important moments.

The conditional mean and the mean of the DGSV model are:

\begin{equation}
E[H_t|\mathcal{F}_{t-1}] = \mu + \alpha H_{t-1} + \rho \sigma e^{\nu_{t-1} + \kappa^2/2}, \quad (15a)
\end{equation}

\begin{equation}
E[H_t] = \mu + b_0 \rho \sigma \epsilon \frac{\sum_{j=0}^{\infty} \beta_j^{j/2}}{1 - \alpha}. \quad (15b)
\end{equation}

Here $\kappa = \kappa_{t-1} = \beta_0 = b_0$ but since it is a constant, we do not need the time parameter.

The value of the conditional variance and the variance is:

\begin{equation}
D^2(H_t|\mathcal{F}_{t-1}) = \sigma^2 e^{2\nu_{t-1} + \kappa^2} \left( e^{\kappa^2} - \kappa^2 \rho^2 + 4 \kappa^2 \rho^2 e^{\kappa^2} \right), \quad (16a)
\end{equation}

\begin{equation}
D^2(H_t) = \frac{\mu^2 + \sigma^2 e^{2\nu_{0}} \sum_{j=0}^{\infty} \sigma_j^2 + 2(\mu \alpha + R(\alpha M + \mu)) - M^2}{1 - \alpha^2}, \quad (16b)
\end{equation}

where $M := E[H_t]$ and $R := b_0 \rho \sigma \epsilon \sum_{j=0}^{\infty} \sigma_j^{j/2}$.

The value of the conditional second moment is:

\begin{equation}
E[H_t^2|\mathcal{F}_{t-1}] = \mu^2 + \alpha^2 H_{t-1}^2 + \sigma^2 (1 + 4 \rho^2 \kappa^2) e^{2\nu_{t-1} + 2\kappa^2} + 2 \left( \mu \alpha H_{t-1} + \kappa \rho \sigma e^{\nu_{t-1} + \kappa^2/2} + \alpha H_{t-1} \kappa \rho \sigma e^{\nu_{t-1} + \kappa^2/2} \right). \quad (17)
\end{equation}
These moments are important not only because typically it is impossible to calculate them in a stock price model, but they also play an important role when we are using the approximated strategies mentioned in Section 3. In Section 6, we will see that these conditional moments in the DGSV model give a very good approximation of the log-optimal strategy, in contrast to other typical stock price models.

In the previous section we defined the $0-1$ barrier in (11). The For $\alpha > 0$, the function $\tilde{h}$ appearing in the equation of the $0-1$ barrier of this model is:

$$\tilde{h}_{\text{DGSV}}(\nu, \kappa) = -\frac{\mu + \sigma \rho \kappa e^{\nu + \kappa^2/2}}{\alpha}. \tag{19}$$

Remark 7. Unconditional moments only depend on the sum $\sum_{j=0}^{\infty} \beta_j^2$ and $\beta_0$ but not specifically on the decay of the $\beta_j$’s.

It is possible to calculate unconditional skewness and kurtosis but we focus now on conditional skewness and kurtosis because they can help in choosing a realistic parametrization for numerical simulations. Let us denote the conditional skewness and kurtosis by $\text{skew}_{t-1}(H_t)$ and $\text{kurt}_{t-1}(H_t)$ (condition of $F_{t-1}$). Now let us define another log-return model $\tilde{H}_t$ as:

$$\tilde{H}_t = \sigma e^{b_0 \epsilon_t} (\rho \epsilon_t + \sqrt{1 - \rho^2} \eta_t), \tag{20}$$

where the parameters are in common with the DGSV model. Then unconditional skewness and kurtosis are the same for $\tilde{H}_t$ as for the conditional skewness and kurtosis in the DGSV model:

$$\text{skew}_{t-1}(H_t) = \mathbb{E}\left[ \left( \frac{e^{\kappa \epsilon-t/2}(\rho \epsilon + \sqrt{1 - \rho^2} \eta) - \kappa \rho}{\sqrt{e^{\kappa \epsilon} - \kappa^2 \rho^2 + 4 \kappa^2 \rho^2 e^{\kappa \epsilon}}}, \tag{21a} \right) \right],$$

$$\text{kurt}_{t-1}(H_t) = \mathbb{E}\left[ \left( \frac{e^{\kappa \epsilon-t/2}(\rho \epsilon + \sqrt{1 - \rho^2} \eta) - \kappa \rho}{\sqrt{e^{\kappa \epsilon} - \kappa^2 \rho^2 + 4 \kappa^2 \rho^2 e^{\kappa \epsilon}}}, \tag{21b} \right) \right].$$

Remark 8. (21a) and (21b) show that conditional skewness and kurtosis only depend on $b_0$ and $\rho$.

Fitting the parameters of the model would require a whole study in itself and it is also out of the scope of this paper. Despite this, in Section 6, we give a heuristic argument for our choice of parameters in order to make our choice more clear. Taking data from real stock prices we can estimate their log-return’s skewness and kurtosis. Then we are able to choose $b_0$ and $\rho$ such that the skewness and kurtosis of $\tilde{H}_t$ fit the empirical values. That is, by ‘approximating’ our model $H_t$ with $\tilde{H}_t$ we can heuristically choose these parameters. Figure 1 shows these conditional moments of $H_t$ (or moments of $\tilde{H}_t$) as a function of $b_0$ and $\rho$. According to real stock price data the skewness is $< 0$. This is also true in our model when $\rho < 0$, which means skewness of the log-return is a consequence of the leverage effect.
Figure 1. Conditional skewness and kurtosis of log-return in the DGSV model with parameters set to $\sigma = 1, \nu = 0$. Values around $\rho \approx -0.4$ and $b_0 \approx 0.25$ give similar skewness and kurtosis to those that can be observed in typical stock returns.

Table 1. Statistical properties of log-returns investigated in 1250 stocks from New York Stock Exchange.

<table>
<thead>
<tr>
<th>Statistics</th>
<th>Average</th>
<th>(0.05, 0.95) quantile</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.00014</td>
<td>(-0.00074, 0.00081)</td>
</tr>
<tr>
<td>St. dev.</td>
<td>0.02063</td>
<td>(0.00798, 0.04100)</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.65034</td>
<td>(-2.44097, 0.54823)</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>41.3726</td>
<td>(5.19164, 157.17040)</td>
</tr>
</tbody>
</table>

Table 2. Simulation parameters (top) and the statistical results of one realization (bottom) in the two models DGSV and BARCH.

<table>
<thead>
<tr>
<th></th>
<th>DGSV-1</th>
<th>DGSV-2</th>
<th>BARCH-1 – moderate memory</th>
<th>BARCH-2 – long memory</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Params</strong></td>
<td><strong>\mu</strong></td>
<td><strong>a</strong></td>
<td><strong>b</strong></td>
<td><strong>b_0</strong></td>
</tr>
<tr>
<td><strong>Value</strong></td>
<td>1/550</td>
<td>0.3</td>
<td>0.6</td>
<td>1/3</td>
</tr>
<tr>
<td><strong>Value</strong></td>
<td>1/550</td>
<td>0.05</td>
<td>0.6</td>
<td>0.55</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Stat.</strong></td>
<td>DGSV-1</td>
<td>DGSV-2</td>
<td>BARCH-1 – moderate memory</td>
<td>BARCH-2 – long memory</td>
</tr>
<tr>
<td>Mean</td>
<td>-0.00023</td>
<td>0.00006</td>
<td>0.00022</td>
<td>0.00025</td>
</tr>
<tr>
<td>St. dev.</td>
<td>0.017</td>
<td>0.025</td>
<td>0.011</td>
<td>0.016</td>
</tr>
<tr>
<td>Skew.</td>
<td>-1.39</td>
<td>-0.68</td>
<td>-0.66</td>
<td>0.05</td>
</tr>
<tr>
<td>Kurt.</td>
<td>10.0</td>
<td>28.2</td>
<td>19.3</td>
<td>9.3</td>
</tr>
</tbody>
</table>
(i.e., negative correlation between the log-return and the volatility). In the numerical investigations we chose $(b_0, \rho) = (1/3, -0.4)$ and $(0.55, -0.15)$.

Table 1 shows data from the New York Stock Exchange (NYSE) as a benchmark for our models. The constant $\sigma$ can be chosen to be close to the standard deviation of typical log-returns on NYSE.

Of course, our aim here is not to calibrate the moments of the models, only to give some properties of the model. We will see in Subsection 4.3 that these heuristically chosen parameters give statistically plausible stock price realizations (Table 2).

### 4.2. Bilinear ARCH

Log-returns are more easily accessible than volatilities – this is one of the main reasons conditional heteroscedasticity models became popular. In these models, to forecast the unknown return of the price, investors only have to know past data of the returns and estimate model parameters. The most representative feature of stock prices, mentioned in Section 1, is that although log-return has small auto-correlation, its absolute value or its powers do show long memory. While ARCH or GARCH cannot capture long memory, Linear ARCH and Bilinear ARCH processes can. Not only long memory is captured but anti-persistence as well.

The equations for the Bilinear ARCH process of Giraitis and Surgailis (2002) are the following:

\[
H_t = c_0 + \sum_{j=1}^{\infty} c_j H_{t-j} + \eta_t \sigma_t, \quad t \geq 0, \tag{22a}
\]

\[
\sigma_t = a + \sum_{j=1}^{\infty} \beta_j H_{t-j}, \tag{22b}
\]

where $a, c_j, \beta_j$ are real parameters. If we choose $c_j = 0$ for all $j \geq 0$, then we get the Linear ARCH model. The term $\sigma_t$ is the conditional variance, while $c_0 + \sum_{j=1}^{\infty} c_j H_{t-j}$ is the conditional mean. With the proper choice of these parameters both $H_t$ and $\sigma_t$ will be ergodic processes with long memory. Recalling the observation that memory is carried within the volatility and not in the drift (Gatheral, Jaisson, and Rosenbaum (2018)), we choose coefficients $c_j = 0$ for all $j \geq 2$. These choices keep the conditional mean part simple. For the sake of similarity, we are going to denote the parameters $c_0$ and $c_1$ as $\mu$ and $\alpha$. We imagine $c_1 = \alpha$ is small, because larger $\alpha$ values result in a stronger auto-correlation of the log-return, which should be negligible. Nevertheless, it is more difficult to find a proper parametrization of the models if $\alpha$ has a low value. For these reasons we concluded we should use $\alpha = 0.3$ in the numerical simulations (we indicate where we change this setting).

In the notation of Section 2 we can ignore the noise $\varepsilon$ as well as $Y_t$, so $H_t = F(Z_{t-1}, \eta_t)$, where $Z_{t-1} = (H_{t-1}, H_{t-2}, \ldots)$ is an infinite length vector of past log-returns. The function
is \( F(z, \eta) = \mu + \alpha z_1 + (a + \beta^T z) \eta \), where \( \beta = b_0(2^{-b}, 3^{-b}, 4^{-b}, \ldots) \) and \( z \) are the infinite vectors and \( z_1 \) is the first component of \( z \).

We take \( \sigma_t = a + b_0 \sum_{j=1}^{\infty} j^{-b} H_{t-j} \) so we finally get this equation:

\[
H_t = \mu + \alpha H_{t-1} + \left( a + b_0 \sum_{j=1}^{\infty} j^{-b} H_{t-j} \right) \eta_t, 
\]

where \( \eta_t \sim \mathcal{N}(0, 1) \) are i.i.d. variables. White noise \( \eta_t \) could have a different distribution if they are independent and the distribution is known. We chose Gaussian law only for simplicity.

For \( \alpha > 0 \), the formula of the 0–1 barrier in (11) in this case is:

\[
\bar{h} = -\frac{\mu}{\alpha}.
\]

As we can see it is a constant, independent of the structure of the volatility term. In the DGSV model the volatility affects the first order approximation through the leverage effect. Both in the DGSV and the BARCH model we gave the restriction that \( \alpha > 0 \). When \( \alpha < 0 \) the condition in the definition of the 0–1 barrier function changes to \( H_{t-1} < \bar{h}(v_{t-1}, \kappa_{t-1}) \).

### 4.3. Further Statistical Properties of the Models

(Conditional) moments for the DGSV model and some conditional moments for the BARCH model were given in the previous subsections. Here, we would like to show some descriptive statistics with financial meaning based on numerical simulations of stock prices. For this purpose, we compare the results with real financial data. Of course, in this paper we do not aim to fit the parameters of the models, since that would require an entire study, therefore we use heuristically chosen parameters.

We investigated 1250 stock prices of the New York Stock Exchange. The basic descriptive statistics can be found in Table 1.

In the two models we used two different parametrizations. DGSV-1 and DGSV-2 models mainly differ in the mean-reverting parameter \( \alpha \) and the leverage coefficient \( \rho \). This distinction will cause an essential difference in their optimal strategy, as we will see in Section 6. In the BARCH models we used two cases, one with long and one with moderate conditional variance memory. Although we did not prove the ergodicity of \( Q_t \) in this model family, and ergodicity of long memory BARCH is not known as of yet, in our parameter choice they produce fairly similar results.

To see the memory effects clearly, we chose some parameters that allow the statistical data to be outside of Table 1. For example, the constant \( \mu \) in the drift was chosen to be smaller than a more realistic model would suggest. The reason for this is that in the numerical simulations we used \( r = 0 \) for interest rate to ignore the interest rate’s effect. If we increase the constant \( \mu \) we can get a more realistic mean for the log-return and other statistics do not change significantly. Apart from a few outlier statistical data, most of the data are in the expected range based on the stock price data from NYSE; more importantly, the autocorrelation of the log-returns and the absolute log-returns...
are adequate. Clearly, the best fits are from the model DGSV-2 and the BARCH – moderate memory.

We simulated the stock prices for 4000 days; Table 2 shows the parameter settings of the models (top) and the descriptive statistics of simulated log-returns for one realization (bottom). In all four cases we used the same thread of noise. Figures 2 and 3 show the autocorrelation of the log-return and absolute log-return in both models. There is significant autocorrelation only in the absolute log-return as is the experience for typical stock price data. A slight difference in the autoregressive part ($\mu$ and $\alpha$) of the two models is unavoidable because of their different effect on the stock price.

![Figure 2. Autocorrelation of the Discrete Fractional Stochastic Volatility model with parameters shown in Table 2. DGSV-2 model has smaller $\alpha$ parameter which is responsible for the less significant autocorrelation of $H_t$ and it has more significant autocorrelation for $|H_t|$ because the memory effect in the log-volatility is stronger (higher $b_0$ value). Both DGSV models have long memory in the volatility process.](image-url)
Based on simulations the role of the drift and volatility is clear: the larger the mean-reverting coefficient $\alpha$, the more significant the autocorrelation of the log-return $H_t$, while the strength of the memory (higher $b_0$ or lower $b$) is related to the significance of the autocorrelation of the absolute log-return $|H_t|$. Since we are looking for stock price models which describe memory effect well, we have to choose models with lower $\alpha$ and higher $b_0$ or lower $b$.

We can see that values from DGSV and BARCH – short memory models are in the range of the (5%, 95%) quantiles calculated from the New York Stock Exchange, shown in Table 1. It is probable that the distinct result of the BARCH – long memory, which do not correspond to the values could be resolved with fitting its parameters.

Figure 3. Comparison of the moderate and long memory BARCH process with exponent $b = 1.1$ and $b = 0.65$ with their autocorrelation function of log-returns and their absolute value in the Bilinear ARCH model. Parameters are shown in Table 2. BARCH-1 has moderate memory, BARCH-2 has long memory in their volatility process.
There is an immense difference between the moderate and long memory cases of the BARCH model regarding the autocorrelation, although other statistical properties fit well in the moderate memory case. A disadvantage of the Bilinear ARCH model is that it cannot capture real autocorrelation in its volatility which is responsible for the autocorrelation of the absolute log-return. Ergodicity can only be proven for the BARCH – moderate memory.

We used a heuristic argument to choose the parameters $\rho$ and $b_0$ based on the calculation of the conditional skewness and kurtosis (shown on Figure 1). However, the eventual (not conditional) skewness and kurtosis of the chosen models are not equal to the conditional ones, but they are still adequate for our purposes, as they have a good fit.

Mean of the log-return is less than that of real data, the reason for this is that in Section 6 we simulated against $r = 0$ interest rate. For this reason we had to lower the value of $\mu$.

5. Verification of Assumptions

5.1. Gaussian Stochastic Volatility Model

In this subsection we prove that the model presented in Subsection 4.1 has a stationary version that satisfies Assumption 2.6. This justifies to some extent the simulations we have performed for that model. From this assumption it follows, that with optimal strategy $\pi^*$ from Theorem 2.5,

$$\lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \log(W_T^{\pi^*}) \right] = \lim_{T \to \infty} \frac{1}{T} \log(W_T^{\pi^*}).$$

That is, instead of calculating the expected value for every $T$ and taking the limit, it is enough to determine the optimal strategy for only one realization. This optimal strategy is iterated according to (7). In Section 6, where we show numerical results, we are going to see that this convergence is achieved reasonably fast.

Theorem 5.1. There exists a stationary process $\tilde{X}_t$, $t \in \mathbb{Z}$ satisfying

$$\tilde{X}_{t+1} = \mu + \alpha \tilde{X}_t + \rho e^Y_t \eta_{t+1} + \sqrt{1 - \rho^2} e^Y_t \epsilon_{t+1},$$

such that $\tilde{X}_t$ is $\sigma(\eta_j, \epsilon_j, j \geq t)$-measurable. Defining

$$Q_t := g^{\tilde{X}_t, v_t, \kappa_t}(\tilde{X}_t, v_t, \kappa_t),$$

and

$$F(z, y, \epsilon, \eta) := \mu + az + \rho e^Y \eta + \sqrt{1 - \rho^2} e^Y \epsilon,$$

Assumptions 2.2 and 2.6 are valid.

Proof. We follow the arguments of Diaconis and Freedman (1999) closely. Consider the mapping

$$(x, y, \eta, \epsilon) \to L(x, y, \eta, \epsilon) := \mu + ax + \rho e^Y \eta + \sqrt{1 - \rho^2} e^Y \epsilon,$$
where \( x, y, \eta, \varepsilon \in \mathbb{R} \). This satisfies
\[
|L(x_1, y, \eta, \varepsilon) - L(x_2, y, \eta, \varepsilon)| \leq |\alpha||x_1 - x_2|
\]
so it is a contraction. For each \( n \geq 1 \) and for each \( t \in \mathbb{Z} \), define, by recursion,
\[
X^{t,n}_{t-n} := 0, \quad X^{t,n}_{k+1} := L(X^{t,n}_k, Y_{k+1}, \eta_{k+1}, \varepsilon_{k+1}), \quad k \geq t - n.
\]
Set \( \tilde{X}^n_t := X^{t,n}_t \). Since
\[
|\tilde{X}^n_t - \tilde{X}^{n+1}_t| \leq |\alpha|^n |\rho e^{Y_{t-n}} \eta_{t-n} + \sqrt{1 - \rho^2} e^{Y_{t-n}} \varepsilon_{t-n}|,
\]
and the right-hand side is bounded in \( L^1 \), it follows by the Markov inequality that
\[
\sum_{n=1}^{\infty} P\left(|\tilde{X}^n_t - \tilde{X}^{n+1}_t| \geq \sqrt{|\alpha|^n}\right) \leq \sum_{n=1}^{\infty} C \frac{|\alpha|^n}{\sqrt{|\alpha|^n}} < \infty,
\]
with some \( C > 0 \). The Borel–Cantelli lemma implies that the sequence \( \tilde{X}^n_t \), \( n \in \mathbb{N} \) is a.s.a Cauchy sequence, hence it is convergent almost surely to some \( \tilde{X}_t \).

Since \( \tilde{X}^n_t \), \( t \in \mathbb{Z} \) is easily seen to be a stationary process for each \( n \geq 1 \), the process \( \tilde{X}_t \), \( t \in \mathbb{Z} \) is stationary, too. Notice that \( \tilde{X}^{n+1}_{t+1} = L(\tilde{X}^n_t, Y_{t+1}, \eta_{t+1}, \varepsilon_{t+1}) \) and \( L \) is continuous in its first variable so we conclude that (31) indeed holds true. The claim about measurability is clear from the construction of \( \tilde{X} \). Theorem 3.5.8 of Stout (1973) implies that \( \tilde{X} \) is ergodic. Actually, that theorem implies also that \( G_t := (\tilde{X}_t, Y_t, \eta_t, \varepsilon_t), \ t \in \mathbb{Z} \) is ergodic since \( G_t \) is a functional of \( (\varepsilon_t, \eta_t, \varepsilon_{t-1}, \eta_{t-1}, \ldots) \). Furthermore, as \( Q_t \) is a functional of \( (\tilde{X}_t, Y_t, \eta_t, \varepsilon_t) \) it is also ergodic.

We now claim \( \mathbb{E}|Q_t| < \infty \). Notice that, using the Cauchy inequality,
\[
\mathbb{E}\left[|az + \rho e^{\nu+\kappa \theta} \varepsilon_0 + \sqrt{1 - \rho^2} \eta_0|\right] \leq |z| + e^{2\nu+2\kappa^2} \times [1 + \mathbb{E}\eta_0^2]
\]
and the sequence \( e^{2\nu_i} \) is clearly bounded in \( L^1 \). Hence, by Lemma 2.1, it suffices to show that the sequence \( \tilde{X}^n_t \) is bounded in \( L^1 \). Notice that
\[
\mathbb{E}[X^{n,t}_{k+1}] \leq |\alpha| \mathbb{E}[X^n_k] + \mathbb{E}\left[|Y_{k+1}| + |\eta_{k+1}| + |G_{k+1}|\right] \leq |\alpha| \mathbb{E}[X^n_k] + C
\]
for some \( C > 0 \) which easily implies that all the \( X^{n,t}_k \) are bounded in \( L^1 \), which shows our claim.

Now Assumption 2.2 simply holds as
\[
\sup_{z,v,\kappa \in [-N,N]} |F(z, v + \kappa \varepsilon_0, \varepsilon_0, \eta_0)| \leq N + e^{N+|\varepsilon_0|} \times \left[|\varepsilon_0| + |\eta_0|\right]
\]
which clearly has a finite expectation.
5.2. Bilinear Arch-Type Model

Ergodic theorems for the Bilinear ARCH model are proven in Giraitis and Surgailis (2002), here we restrict their result to our simpler setting. Recall the model structure from the previous section for the log-return $H_t$:

$$H_t = \mu + \alpha H_{t-1} + \left( a + b_0 \sum_{j=1}^{\infty} (1+j)^{-b} H_{t-j} \right) \eta_t,$$  

(27)

where $\mu \neq 0$, $\eta_t \sim \mathcal{N}(0,1)$ i.i.d. and $\mu, a, b_0, b$ are real parameters.

The following assumption is a reduced form of the Assumptions A1, A2, A3 and Proposition 2.7 (iii) from Giraitis and Surgailis (2002) in:

Assumption 5.2. In the model (27) let $\mu \neq 0$, $b > 1$, $|a| < 1$ and

$$\left( \frac{b_0}{1 - \alpha} \zeta(2b) \right)^2 < 1.$$  

(28)

The function $\zeta(\cdot)$ is the (Euler–)Riemann zeta function, which is defined as $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$.

Theorem 5.3. Let Assumption 5.2 be satisfied. Then (27) has a unique, strictly stationary and ergodic solution. Moreover, its expected value is $\mathbb{E}[H_t] = \mu/(1 - \alpha)$.

Proof. See Propositions 2.4 and 2.7 of Giraitis and Surgailis (2002). \hfill \Box

Ergodicity of the log-return and its finite first moment is a necessary condition but not sufficient. The proof of the previous model does not work here, because the function $Z(\cdot)$ has infinitely many variables ($H_t, H_{t-1}, \ldots, \eta_{t+1}$). Nevertheless, numerical results suggests that ergodicity of $H_t$ in the BARCH model is also sufficient.

Remark 9. As it was mentioned before in Section 4, this model cannot provide ergodicity in the long memory case in contrast to the DGSV model.

Assumptions in Giraitis and Surgailis paper Giraitis and Surgailis (2002) are very complicated regarding a general Bilinear ARCH process. Our simplified version gives a more reasonable condition on the parameters as we can see in (28), but visualization is still desirable. Figure 4 shows this restriction in a practical way.

6. Numerical Results

In this section we show some numerical results on the log-optimal solution and its approximations. As we have mentioned in Section 5, the long horizon optimization problem in (5) can be solved going along only one realization (see (24)), concluded from Assumption 2.6. Therefore, we can denote the long horizon value function as $G^* := \lim_{T \to \infty} \log(W_T^*)/T$. Numerical simulations were performed until the finite time horizon $T = 4000$, its value function is denoted as $\log(W_T^*)/T$. (It should be noted, that
this log($W_T$)/$T$ is not the solution of the short horizon problem in (4), but the optimizer strategy is the same until $T$). We recall here that the optimal strategy function (7) is based on the function $	ilde{\pi}(z, v, \kappa)$:

$$
\pi^*_t = \tilde{\pi}(Z_{t-1}, v_{t-1}, \kappa).
$$

(29)

It is worth comparing the outcomes of the log-optimal strategy with that of another, well-known strategy, and also with the approximative strategies described in Section 3. For this purpose, we chose the constantly rebalanced portfolio as a frequently used strategy, where the investors use a constant strategy function

$$
\pi_t = \pi^{reb} \in [0, 1].
$$

If the log-price process has independent increments, then this strategy is also log-optimal. In our models, which are more complex, it is suboptimal.

The best constantly rebalanced investor is one who optimizes the expected value function: $\pi^{reb*} := \arg \max_{\pi^{reb} \in [0, 1]} \{\lim_{T \to \infty} \mathbb{E}[\log(W_T^{\pi^{reb}})/T]\}$. In practice we determine this strategy by Monte-Carlo simulation of the models. The difference $\Delta_T := (\log(W_T^*) - \log(W_T^{\pi^{reb*}}))/T$ between the log-optimal and the best constantly rebalanced portfolio gives the measure of how much better the log-optimal strategy performs.

The convergence in Assumption 2.6 works for the approximative strategies and the constantly rebalanced strategy as well, therefore, the value function converges to a constant in every case. Let us denote the value function by $G^*$, $G^{lin}$, $G^{quad}$ and $G^{reb}$ respectively for the log-optimal, the first and second order approximative and the constantly rebalanced strategies.
To better understand the value function of each strategy, which can be considered as a daily compound rate, we convert the value function to annual yield (\( AY \)) or yearly interest rate, which equals \( \exp(250G)/C_0 \); that is, how much the portfolio's value grows in one year with value function (growth-rate) \( G \).

Of course, in the numerical results we use a finite \( T \) to approximate the growth-rate. Because it is a random variable, the mean of the annual yield in the numerical simulations is relevant. Here we call this the mean annual yield (\( MAY \)). For example, for the log-optimal investor its mean annual yield is

\[
MAY^* := \sum_i e^{250 \log(W_t^*(i))} - 1,
\]

where \( i \) denotes the \( i \)th realization (of which we took 100) and \( T = 4000 \) is the number of time-steps in the simulation.

In every figure we used the same parameter settings as in Table 2. Whenever we changed the parameter, it is explicitly indicated. In every figure we used the same
realization(s) to allow us to compare the effect of the parameters. We refer to different realizations as seeds.

6.1. Log-Optimal Solution

The two most important questions to ask about a new optimization algorithm are: (1) Is the optimum achieved on a ‘reasonable time horizon’? (2) Is this optimum essentially better than competing strategies like constant rebalanced portfolios? The answer for the first question is shown in Figure 5. The value function becomes stable around $t=1000$ in the DGSV model and around $t=1500$ in the BARCH model, which is about 4–6 years assuming daily portfolio rebalanced. It is reasonable, since log-optimal investments are typically used for long term portfolios. (In our simulation $T=4000$, i.e., roughly 16 years.)

Value function and yearly interest rates are very high in the results of the simulation. Clearly, transaction fees of all types are excluded, which is why such excellent returns can be achieved.

The other important question is answered in Figure 6. With the same parameter settings as in Table 2, this figure shows how the log-optimal solution exceeds the best constantly rebalanced strategy for 100 seeds. The results are consistent and different driving noise did not affect them. We remind the reader that the two main differences between DGSV-1 and DGSV-2 models is that the latter one has smaller autoregressice coefficient ($\alpha$) and stronger memory ($b_0$ is higher). The result of this difference is that the convergence of the DGSV-2 is slower, hence the higher deviation on the figure.
Surprisingly, the BARCH-2 model with long memory also shows convergence, despite being unable to prove any kind of ergodicity (and Assumption 2.6 is also not proved). The BARCH-1 model has lower standard deviation than the others, which is obvious, since its memory is only of moderate type.

The variable $\Delta_T$ shows how the log-optimal investor overcomes the constantly rebalanced one. In the case of DGSV-2 we see, that $T = 4000$ was not enough long to beat the benchmark strategy in 5 out of 100 realizations (we note that, by the ergodic assumption, $\Delta_T$ converges to a positive constant a.s.).

**Figure 7.** Relation between the value function of the two DGSV and the two BARCH models and the memory decay parameter $b$. The parameters are identical to the ones in Table 2 except for the exponent $b$. The model BARCH is not ergodic on every value of $b > 0.5$ unlike the model DGSV, therefore a dashed vertical line shows where the BARCH model is ergodic.

**Figure 8.** We show how the parameter $b_0$ affects the value function when the term $\sum \beta_j^2$ is kept constant. It appears that their relationship is linear, as we show with a fitted dashed black line on the five different realizations at once.
6.2. Analysing the Role of Memory

Figures 7 and 8 investigate the relationship between the strength of the memory and the performance of the log-optimal investor for five different scenarios (different random sequences). Regarding the strength of the memory, we are focusing on the exponent $b$, which is the decay of the past information. We remind the reader that when $b \leq 1$, the model has long memory, and when $b > 1$, it has moderate memory (both have infinite memory, but the former one is stronger).

Figure 7 shows the value function for the four models described in Subsection 4.3 but for different $b$ values. While in the previous sections the model BARCH-1 referred to the moderate memory case and BARCH-2 to the long memory, here it is different, since $b$ determines the type of the memory. Interestingly, in those cases where the ergodicity assumption can be proven for the models, the value function is independent of $b$ and the value function of different realizations are close to each other. We can see this on Figure 7. In the cases of DGSV-1/2 when $b \approx 0.5$, this effect deviates a little. In the long memory BARCH case ($b < 1$) the relationship between the value function and the memory decay parameter $b$ cannot be described.

We have seen in Section 4 that every unconditional moments of the DGSV model only depends on the memory through $b_0$ and $\sum \beta_j^2$ (where $\beta_j = b_0(1 + j)^{-b}$). Based on this, we wanted to investigate how the value function depend on $b_0$ while the sum $\sum \beta_j^2$ is kept constant. We found that the value function has an approximately linear dependency structure on $b_0$. Figure 8 displays their relation on a log-log plot and shows a linear relationship between them. Since the linear fit on the data (black line) performs well, it suggests a power law dependency between the value function and the decay parameter.

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Figure 9. Comparing the Value function of the constantly rebalanced investors (left) versus the log-optimal investor (right, histogram of their strategy). The parameters are the same as in Table 2, DGSV-1 model. The best constantly rebalanced portfolio was determined by MC simulation of 200 paths with $T = 4000$. The value function reaches its maximum when we invest 60% in the risky asset. The yearly average yield of the constantly rebalanced and the log-optimal investor is: 1.5% and 66%. The log-optimal investor mostly uses 0 or 1 strategy.

\[^{3}\text{In the DGSV model the term } \sum \beta_j^2 \text{ appears as the log-volatility. The formulation of this term in continuous time could be formulated as the variance of the log-volatility.}\]
As an interesting fact we mention that the expected value of the log-return in model DGSV is a linear function of \( b \), and in the continuous time-model the value function of the log-optimal investor is also a linear function of the mean. Unfortunately there is no evidence that it is indeed true in the discrete case (or at least in our case).

6.3. Approximation of the Log-Optimal Strategy

In the numerical simulations we found that log-optimal strategy is extreme in the sense that the investor only buys a riskless or risky asset every time but hardly uses mixed strategies (Figure 9). In a negligible part of the time there can be mixed decisions but this is so rare that it cannot even be seen on the histogram. In the DGSV-1 parameter settings the log-optimal investor used mixed strategy only 61 times out of the 4000 investment days, while in the DGSV-2 case the log-optimal investor used mixed strategy 338 times out of the 4000 investment days. The reason for this is that in the second case the autoregressive coefficient \( \alpha \) is smaller (0.05 instead of 0.3) and that the memory is stronger \( (b_0 = 0.55 \text{ instead of } 1/3) \), which results in a much higher volatility.

We have seen that the optimal strategy \( \pi_t^* \) is determined by the optimal selector function \( \hat{\pi}_t(h, \nu, \kappa) \) as (7) shows and the approximated portfolios approximate this log-optimal strategy. The first order approximation from Section 3 answers the question of why the log-optimal strategy is mostly 0 or 1. Since the standard deviation of the log-return is around 0.02, even the first order approximation performs well where the optimal strategy is exclusively 0 or 1 (see (11)). Figure 10(a) shows the log-optimal strategy selector function \( \hat{\pi} \) for the DGSV-2 model on a blue coloured scale, while the red curve on the figure shows the 0–1 barrier from (19). The first order approximated strategy is 1 above the red curve and 0 under it. We can see that the red curve approximates well where the log-optimal strategy is closer to 0 or 1.

Figure 10(a) shows a small region on the plane \( (h, \nu) \) where the log-optimal investor uses mixed (between 0 and 1) decisions. The second order approximation explains this

![Figure 10. Log-optimal strategy and its approximations. Figure (a) compares the log-optimal strategy with the first order approximation (Lin. approx). Above the red curve the approximated strategy is 1, and below it, it is 0. Figure (b) compares the log-optimal strategy with the quadratic approximation. Instead of the whole \( \hat{\pi}(h, \nu) \) two-variable function, we only show results for high values of \( \nu \), where the difference between the two strategies is more significant. Nevertheless, it is difficult to see the difference between the strategies on the plots.](image-url)
deficiency and Figure 10(b) shows that this approximation is almost impossible to distinguish from the log-optimal one on the figure. The size of the mixed strategy region is proportional to the memory. The higher the $b_0$ or smaller the $b$ or smaller the $\alpha$, the bigger this region is.

Of course, the approximated strategies cannot be the same as the log-optimal one. In the model DGSV-1, where the autoregressive part is also significant ($\alpha = 0.3, b_0 = 1/3$), the approximations perform well, while in the DGSV-2 model, where $\alpha = 0.05, b_0 = 0.55$, the mixed strategy region is bigger, hence the approximated strategies perform more poorly.

Numerical results of the competing strategies are displayed in Table 3. We observed that the quadratic approximation performs well in each case. In fact, in the case of BARCH-2, the approximation exceeds expectations (where log-optimal solution is not proved). The reason for this good performance is that the quadratic approximation simply uses function evaluation, whereas in the log optimal case, we use numerical integration and we need to discretize the strategy’s interval. Linearly approximated strategy works well, but slightly worse in the case of DGSV-2, where the memory is stronger and the AR coefficient $\alpha$ is small. The best constantly rebalanced investor performs poorly in all cases compared to the log-optimal or the approximated log-optimal cases.

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Mean annual yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>Log-optimal</td>
<td>62.080%</td>
</tr>
<tr>
<td>2nd order approx.</td>
<td>62.075%</td>
</tr>
<tr>
<td>1st order approx.</td>
<td>62.051%</td>
</tr>
<tr>
<td>Best const. rebal.</td>
<td>2.0946%</td>
</tr>
<tr>
<td>DGSV-1</td>
<td>29.641%</td>
</tr>
<tr>
<td>DGSV-2</td>
<td>32.568%</td>
</tr>
<tr>
<td>BARCH-1</td>
<td>70.966%</td>
</tr>
<tr>
<td>BARCH-2</td>
<td>70.971%</td>
</tr>
<tr>
<td>2nd order approx.</td>
<td>29.623%</td>
</tr>
<tr>
<td>1st order approx.</td>
<td>28.384%</td>
</tr>
<tr>
<td>Best const. rebal.</td>
<td>14.053%</td>
</tr>
<tr>
<td>2nd order approx.</td>
<td>32.567%</td>
</tr>
<tr>
<td>1st order approx.</td>
<td>32.569%</td>
</tr>
<tr>
<td>Best const. rebal.</td>
<td>1.618%</td>
</tr>
<tr>
<td>2nd order approx.</td>
<td>70.966%</td>
</tr>
<tr>
<td>1st order approx.</td>
<td>70.971%</td>
</tr>
<tr>
<td>Best const. rebal.</td>
<td>2.673%</td>
</tr>
</tbody>
</table>

### Table 3. Comparing the profitability of different portfolios (different investment strategies) based on simulation of 100 paths for $T = 4000$ investment days. Since the bank’s interest rate $r = 0$, every yearly interest rate above 0 is profitable.

7. **Conclusion**

In summary, numerical simulations show that our algorithm performs well in the models we considered. Not only can we prove that we have indeed found the optimal strategy, we can also evaluate its performance (which would not be possible in the general setting of e.g., Algoet and Cover (1988). The optimal strategy significantly beats constantly rebalanced strategies. Furthermore, the optimal strategy can be calculated in a simple way via numerical integration or further simplified with well performing approximations.

We were able to investigate the effect of the strength of memory on investment success in a quantitative way. We found relationships between parameters related to memory and the value function of the log-optimal investment.

Comparison with real statistical data makes it plausible that it may indeed be profitable to apply this strategy in investment practice. Knowing the underlying model, $\pi$ and $g^\pi$ can be determined in advance and the data can be fed into it as time goes on. The model can easily be extended to include several assets and exogenous macroeconomical factors. In the future, we plan to develop and implement versions of the algorithm that work online, without knowledge of the underlying model and for different utility functions.
Disclosure statement

No potential conflict of interest was reported by the author.

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References


