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Extended formulations of lower-truncated transversal polymatroids

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\textbf{ABSTRACT}

Extended formulations of \((k, l)\)-sparsity matroids defined on graphs with \(n\) vertices and \(m\) edges are investigated by Iwata et al. [Extended formulations for sparsity matroids, Math. Program. 158 (2016), pp. 565–574]. This note interprets results on \((k, l)\)-lower-truncated transversal polymatroids by the first author in 1983, from the viewpoint of extended formulations, and shows the same \(O(mn)\) bound when \(k \geq l\) and a better bound \(O(m^2)\) when \(k < l\). A unified polymatroidal approach is given to derive more general understanding.

\section{1. Introduction}

A polytope \(P\) may be expressed as the projection of a polytope \(Q\) with less facets in higher-dimensional space. The extension complexity \(xc(P)\) of \(P\) is the minimum number of facets of such polytopes. For polymatroids, any linear optimization over them can be solved efficiently by a greedy algorithm, while a certain matroid has exponential extension complexity, as shown by Rothvoß [11] (see also Rothvoß [12] for graph matchings). This poses a problem of investigating a nice class of matroids with polynomial extension complexity.

Martin [9] firstly shows a class of matroids with polynomial extended formulations by reformulating problems with new auxiliary variables as follows. The base polytope of a graphic matroid, for a graph \(G' = (V, E)\) with vertex set \(V\) and edge set \(E\), is shown to have an extended formulation of size \(O(|V|^3)\) \((O(|V||E|)\) as pointed out in [2]). The base polytope of a transversal matroids on \(U\), over a bipartite graph \(G = (U, W; A)\) with left vertex set \(U\), right vertex set \(W\) and edge set \(A \subseteq U \times W\), is shown to have an extended formulation of size \(O(|U||W|)\). Iwata et al. [8] show the base polytope of a \((k, l)\)-sparsity matroid on \(G'\) has extension complexity of \(O(|V||E|)\) when \(k \geq l\), and \(O(|V|^2|E|)\) when \(k < l\) by devising randomized communication protocols as an extension of the protocol in Faenza et al. [2]. For bipartite matchings, the Birkhoff polytope on perfect matchings gives a polynomial-size extended formulation directly [3], which directly implies the above result of transversal matroids.
This note discusses a general framework to regard the sparsity matroid results as special cases from the viewpoint of lower-truncated polymatroids and their derivatives, including \((k, l)\)-lower-truncated transversal matroids, which are defined and algorithmically investigated by Imai [6] (see also [7] and also [10] for polymatroidal treatments on bipartite graphs). First we bound the extension complexity of lower truncation of a general polymatroid. For a \((k, l)\)-lower-truncated transversal polymatroid over \(U\) on bipartite graph \(G = (U, W; A)\) with integer parameters \(k, l\), we show that the extension complexity of its base polytope is \(O(|U||A|)\) in general and \(O(|W||A|)\) when \(k \geq l\). When applied to a \((k, l)\)-sparsity matroid, the bounds are the same for \(k \geq l\), while our bound is better than [8] when \(k < l\). Moreover, our approach explicitly describes extended formulations of these bounds, which may be directly used for linear optimization. These bounds are given in a technical report [5], and this note focuses on the direct descriptions of extended formulations in this general framework.

2. Extension complexity of polymatroids

For a polytope \(P\), another polytope \(Q\) in the same or higher dimensional space is called an extension of \(P\) if \(P\) is derived as a linear projection of \(Q\). The extended complexity \(xc(P)\) of \(P\) is the minimum number of facets of any extension of \(P\).

Edmonds introduced a polymatroid as a polytope in his seminal paper [1] by using the lower truncation from its beginning, and we use his terminology below in this section to pay respect to the paper. A set function \(\rho: 2^E \rightarrow \mathbb{R}\) is a \(\beta_0\)-function if it satisfies the following: (1) \(\rho(X) \geq 0\) for \(\emptyset \neq X \subseteq E\), (2) \(\rho(Y) \leq \rho(X)\) for \(Y \subseteq X \subseteq E\) (monotonicity), (3) \(\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)\) for \(X, Y \subseteq E\) (submodularity). Then, a polytope

\[
P(\rho) = \{x \in \mathbb{R}^E \mid x \geq 0, x(X) \leq \rho(X) \ (\emptyset \neq X \subseteq E)\}
\]

is a polymatroid, where \(x(X) = \sum_{e \in X} x(e)\).

For this polymatroid \(P(\rho)\), consider \(l\) satisfying \(0 \leq l \leq \min\{\rho([e]) \mid e \in E\}\). Then, \(\rho': 2^E \rightarrow \mathbb{R}\) defined by \(\rho'(X) = \rho(X) - l(X \subseteq E)\), \((E, \rho')\) is a \(\beta_0\)-function, which defines a polymatroid \(P(\rho')\). This polymatroid is called an \(l\)-lower-truncated polymatroid obtained from \((E, \rho)\), simply a lower-truncated polymatroid. The membership problem for \(P(\rho')\) can be characterized by using \(P(\rho)\) as follows, where \(\chi_e \in \mathbb{R}^E\) for \(e \in E\) is a unit vector on the underlying set \((E\) here) with \(\chi_e(e) = 1\) and others 0.

Lemma 2.1: \(x \in P(\rho')\) iff \(x + l\chi_e \in P(\rho)\) for each \(e \in E\).

**Proof:** Suppose \(x \in P(\rho')\). For each nonempty \(X \subseteq E\), we have \(x(X) \leq \rho'(X) = \rho(X) - l\), and hence \((x + l\chi_e)(X) \leq x(X) + l \leq \rho(X)\), which implies \(x + l\chi_e \in P(\rho)\).

Conversely, if \(x + l\chi_e \in P(\rho)\) for \(e \in E\), then, for \(X\) with \(e \in X \subseteq E\), we have \(x(X) + l \leq \rho(X)\). This holds for each \(e \in E\), and the lemma follows.

This is an extension of Theorem 2.1 in [6], whose variant is Lemma 3.4 in this note.

Theorem 2.2: \(xc(P(\rho')) \leq |E| \cdot xc(P(\rho))\).
Proof: Suppose that \( P(\rho) \) can be represented as \( \{ x \mid F x + G y = h, \ y \geq 0 \} \), where \( F, G \) are some matrices, \( h \) is some vector, \( x \in \mathbb{R}^F \), \( y \in \mathbb{R}^d \) with \( d = xc(P(\rho)) \). Then, \( P(\rho') \) can be expressed by \( \{ x \mid F x + G y^{(e)} = h - F \cdot I_{X_e}, \ y^{(e)} \geq 0 \ (e \in E) \} \), with introducing independent \( y^{(e)} \) for each \( e \in E \). In the expression, the total number of inequalities is \( d|E| \).

Deletion/contraction of an element and truncation with respect to some vector \( a \in \mathbb{R}^d \) for polymatroid \( P(\rho) \) are simpler operations than lower truncations, and yield polymatroids whose extension complexity is \( O(xc(P(\rho)) + |E|) \).

### 3. Network flow and lower-truncated transversal polymatroids

Let \( \tilde{N} = (\tilde{V}, \tilde{A}; S, t, \tilde{c}) \) be a network with a vertex set \( \tilde{V} \), a directed edge set \( \tilde{A} \), a set of sources \( S \subset \tilde{V} \), a unique sink \( t \in \tilde{V} - S \), and a capacity \( \tilde{c} \in \mathbb{R}^{\tilde{A}} \) where \( \tilde{c}(a)(\geq 0) \) is a capacity of \( a \in \tilde{A} \). For \( f \in \mathbb{R}^{\tilde{A}} \), we define \( \partial^+ f \in \mathbb{R}^{\tilde{V}} \) by \( \partial^+ f(v) = -\sum_{e=(a,v)\in\tilde{A}} \tilde{c}(a) + \sum_{e=(v,w)\in\tilde{A}} f(e) \) for \( v \in \tilde{V} \). The restriction of \( \partial^+ f \) to a subdomain \( X \subseteq \tilde{V} \) is denoted by \( \partial^+ f|_X \). \( f \in \mathbb{R}^{\tilde{A}} \) is a flow if \( 0 \leq f(a) \leq \tilde{c}(a) (a \in \tilde{A}) \), \( \partial^+ f|_{V-(S\cup\{t\})} = 0 \) and \( \partial^+ f|_S \geq 0 \).

A cut function \( \tilde{c}: 2^{\tilde{V}-\{t\}} \rightarrow \mathbb{R} \) is defined by \( \tilde{c}(Y) = \sum_{e=(u,w)\in\tilde{A},u\in Y,w\in\tilde{V}-Y} \tilde{c}(e) \) for \( Y \subseteq \tilde{V} - \{t\} \). \( \gamma_{\tilde{N}}: 2^S \rightarrow \mathbb{R} \) is defined to be \( \gamma_{\tilde{N}}(X) = \min\{\tilde{c}(Y) \mid Y \subseteq \tilde{V} - \{t\}, Y \cap S = X\} \) for \( X \subseteq S \). \( (S, \gamma_{\tilde{N}}) \) is a polymatroid and the following is well known.

**Lemma 3.1:** For \( x \in \mathbb{R}^S \), \( x \in P(\gamma_{\tilde{N}}) \) iff there is a flow \( f \) with \( x = \partial^+ f|_S \). Hence \( xc(P(\gamma_{\tilde{N}})) \leq 2|\tilde{A}| + |S| \).

Let \( G = (U, W; A) \) be a bipartite graph with left vertex set \( U \), right vertex set \( W \) and directed edge set \( A \subseteq U \times W \) where edges are directed from \( U \) to \( W \). By adding a new vertex \( t \) with new directed edges \( \{(w, t) \mid w \in W\} \) and setting a capacity \( c \) on the new directed edge set \( A' \), we derive a network \( N = (U \cup W \cup \{t\}, A'; U, t; c) \) and we denote the integer-valued set function over \( 2^U \) for network \( N \) by \( \gamma_N \), as \( \gamma_{\tilde{N}} \) is defined for \( \tilde{N} \).

In a bipartite graph \( G \), define \( \Gamma(X) \) to be \( \{w \mid (u, w) \in A, \ u \in X\} (X \subseteq U) \). Hall’s theorem states that \( X \subseteq U \) is covered by a matching iff \( |Y| \leq |\Gamma(Y)| \) for all \( Y \subseteq X \). In the network \( N \), we set the capacity \( c \) by \( c(e) = +\infty \ (e \in A) \) and \( c(e) = 1 \ (e = (w, t), w \in W) \), then \( |\Gamma(X)| = \gamma_N(X) \) holds. \( (U, \gamma_N) \) is a transversal polymatroid, and its restriction by \( 1_U \) is a transversal matroid over \( U \) in \( G \). By applying Lemma 3.1 with eliminating some redundant capacity constraints from the network structure, we have the following.

**Lemma 3.2:** \( xc(P(\gamma_N)) = |A| + |W| \). For the transversal matroid over \( U \) of bipartite graph \( G \), its independence polytope has extension complexity of \( |A| + |U| + |W| \).

For \( G = (U, W; A) \), let \( k, l \) be positive integers with \( d'k - l > 0 \), where \( d' \) is the minimum degree of a vertex in \( U \). Define \( \gamma_{k,l}: 2^U \rightarrow \mathbb{R} \) by \( \gamma_{k,l}(X) = k\gamma_N(X) - l (X \subseteq U) \). \( (U, \gamma_{k,l}) \) is a polymatroid, called a \( (k, l) \)-lower-truncated transversal polymatroid. This is an integral polymatroid, and its truncation with respect to \( 1_U \) is called a lower-truncated transversal matroid [6]. We denote this matroid by \( M(G; k, l) \) (denoted simply by \( M(k, l) \) in [6]). Note that \( M(G; 1, 0) \) is a transversal matroid on \( U \), as investigated above. Note that
a class of transversal matroids includes both uniform matroids and partition matroids, and results below apply to those fundamental matroids.

Combining Theorem 2.2 and Lemma 3.2, we have the following.

**Theorem 3.3:** $\text{xc}(P(\gamma_{k,l})) \leq |U|(|A| + |W|)$.

This theorem can also be obtained more or less directly by using Theorem 2.1 in [6], which was used to solve greedy-type optimization problem for the lower-truncated transversal polymatroids. When $k \geq l$, this can be modified as in Theorem 2.5 in [6] by using $W$ instead of $U$ in its bipartite structure. For the network $N = (U \cup W \cup \{t\}, A'; U, t; c)$, we replace $c$ with a new capacity $c_w$ for $w \in W$ defined by $c_w(e) = +\infty$ $(e \in A)$, $c_w(e) = k$ $(e = (w', t), w' \in W - \{w\})$, and $c_w(e) = k - l$ $(e = (w, t))$.

**Lemma 3.4 (Theorem 2.5 in [6]):** For $x \in \mathbb{R}^U$, $x \in P(\gamma_{k,l})$ iff there is a flow $f$ with $x = \partial^+f|_S$ in network $N_w = (U \cup W \cup \{t\}, A'; U, t; c_w)$ for each $w \in W$.

Using this lemma in this case, we may have a better upper bound.

**Theorem 3.5:** When $k \geq l$, $\text{xc}(P(\gamma_{k,l})) \leq |W|(|A| + |W|)$.

Summarizing these results for the lower-truncated transversal matroid, we have only to add $|U|$ inequalities of $x \leq 1_U$ to the extended formulations.

**Theorem 3.6:** For lower-truncated transversal matroid $M(G; k, l)$, its independence polytope $\text{poly}(M(G; k, l))$ has extended complexity (1) $|U|(|A| + |W| + 1)$ in general, and (2) $|U| + |W|(|A| + |W|)$ when $k \geq l$.

### 4. Sparsity matroids

Consider a case of $|\Gamma(\{u\})| = 2$ ($u \in U$) in the bipartite graph $G$. Let $G' = (V, E)$ be an undirected graph with vertex set $V = W$ and edge set $E = U$. $G'$ can be regarded as a graph obtained from $G$ by subdividing each edge $e$ of $G'$ by a vertex $e$ of $G$. The set $V(X)$ of vertices incident to edges in $X \subseteq E$ in $G'$ is equal to $\Gamma(X)$ in $G$. For $G'$ and positive integers $k$, $l$ with $2k - l > 0$, $\{X \mid |Y| \leq k|V(X)| - l (\emptyset \neq Y \subseteq X)\}$ is the set of independent sets of a matroid, called $(k, l)$-sparsity matroid [13] or count matroid [4], which is isomorphic to $M(G; k, l)$ for $G$. $M(G; 1, 1)$ is a graphic matroid of $G$, $M(G; k, k)$ is the union of $k$ identical graphic matroids of $G'$, and $M(G; 2, 3)$ is the rigidity matroid of $G'$. Restating Theorem 3.6, we have the following.

**Theorem 4.1:** For the independence polytope of a $(k, l)$-sparsity matroid of a graph $G' = (V, E)$, there is an extended formulation of size $|E|(2|E| + |V| + 1)$ in general and that of size $|E| + |V|(2|E| + |V|)$ when $k \geq l$.

Applying this theorem, we see that $M(G; k, k)$ has extension complexity of $O(|V||E|)$ and $M(G; 2, 3)$ has extension complexity of $O(|E|^2)$, which improves $O(|V|^2|E|)$ bound in [8].
5. Concluding remarks

In this note we expand a class of matroids whose polytope has a polynomial-size extension formulation, basically utilizing bipartite and network structures underlying the class. This extends to polymatroids and its lower-truncated ones defined by network flow. It would be an interesting problem to investigate the extension complexity for linear matroids.

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