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The Application of Non-Linear Normal Mode Initialization to an Operational Forecast Model

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ABSTRACT The non-linear normal mode initialization technique used in shallow water equation models by Baer (1977) and Machenhauer (1977) is now applied to a full baroclinic primitive equations forecast model. The initialization procedure is shown to be capable of completely removing high frequency oscillations from model integrations, even in the presence of topography. The procedure also produces a consistent and physically realistic initial vertical motion field.

RÉSUMÉ La méthode d'initialisation non-linéaire par modes normaux, appliquée à des modèles aux équations d'une couche peu profonde par Baer (1977) et Machenhauer (1977), est généralisée pour un modèle barocline à équations primitives. Cette procédure d'initialisation s'avère capable d'éliminer presque totalement les modes à hautes fréquences présents durant l'intégration du modèle, et ceci même en présence du relief. En plus, la méthode a l'avantage de produire des champs de courants verticaux qui sont réalistes et compatibles avec tous les autres champs du modèle.

1 Introduction
Numerical models based on the baroclinic primitive equations are the basic tools used to produce today's large-scale weather forecasts. Despite considerable improvement in observational networks and analysis procedures, the initial fields provided for starting-up these models are not fully satisfactory. For example, the initial wind and mass fields are not completely balanced, causing high frequency oscillations in subsequent model integrations. Another problem is that it is difficult to measure the divergent part of the wind

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very accurately and thus the initial vertical motion fields provided to the models are unreliable – resulting in very poor short-term precipitation forecasts.

Machenhauer (1977) and Baer (1977) have recently demonstrated a promising and powerful technique for the initialization of primitive equations models. Non-linear normal mode initialization, as this technique is usually referred to, holds the promise of completely eliminating high frequency oscillations from primitive equations model integrations, even in the presence of non-linear and forcing terms. Both Machenhauer (1977) and Baer (1977) convincingly demonstrated the technique’s effectiveness for the shallow water equations, but its application to the much more interesting and important baroclinic case is only just now being realized (Machenhauer, 1978)*.

The essence of normal mode initialization procedures, both the non-linear form mentioned above, and the earlier linear form (Flattery, 1970, 1972; Williamson, 1976; Williamson and Dickinson, 1976), is the determination of the eigenmodes of a linearized version of the model to be integrated and a characterization of those modes on the basis of their eigenfrequencies. In general, the eigenmodes can be divided into low frequency Rossby modes and higher frequency gravity modes. The normal mode initialization procedures consist in adjusting the original initial data such that the undesirable high frequency oscillations are absent in subsequent model integrations using the adjusted data.

In linear normal mode initialization, this adjustment simply consists of orthogonalizing the original data to the undesirable eigenmodes. This linear technique is not completely successful with non-linear models because the non-linear terms act as forcing terms to regenerate high frequency oscillations during the model integration.

This difficulty led Machenhauer (1977) and Baer (1977) to the non-linear normal mode technique, in which an adjustment is performed such that the time-tendencies of the undesired high frequency modes of the model are initially zero. This can be simply accomplished by projecting the non-linear (and forcing) terms of the model onto the time-tendencies of the undesirable modes and then adjusting the initial amplitudes of those modes, so that the linear contribution to their tendencies exactly cancels the non-linear contribution. This procedure will be successful, in principle, provided that the characteristic time-scale of the non-linear terms is long compared with that of the modes which are to be suppressed from the model integration. This non-linear normal mode procedure was further investigated by Baer and Tribbia (1977) and Leith (1978).

Daley (1978b), in an experiment performed with a shallow water equations model, demonstrated that the non-linear normal mode initialization could be

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put into a variational context to provide direct forced adjustment. This procedure allowed for a non-linear adjustment of the wind and mass fields based on the presumed accuracy of the analyses.

The present work concerns the application of the non-linear normal mode initialization procedure to a particular forecast model based on the baroclinic primitive equations – the Canadian Operational Spectral Model (Daley et al., 1976). This model is a hemispheric sigma co-ordinate model with a horizontal discretization based on spherical harmonics, a semi-implicit time discretization and simple physical parameterizations such as surface drag, topography, moist convective adjustment and large-scale latent heat release. The vertical discretization used in the present study is the variant proposed by Staniforth and Daley (1977) – referred to below as SD – in which the dependent variables are vertically expanded in linear finite elements in sigma.

The equations of the model are first linearized and the normal modes of the linearized equations, found. Particular attention is paid to the vertical structure of the normal modes to ensure that they are derived in a manner completely consistent with the vertical discretization of the model. For the horizontal structure of the normal modes, use was made of the procedure developed for the shallow-water equations model in Daley (1978b) – referred to as D78 below – because its horizontal discretization was identical to that of the present model.

Once the normal modes of the linearized model have been found, they can be used for non-linear normal mode initialization in the manner outlined earlier. The present work is not an attempt to produce an optimal initialization scheme for operational use, but is intended as an exploration of some of the properties of the non-linear normal mode initialization scheme when applied to a baroclinic model. Thus the variational generalization developed in D78 for the shallow-water equations is not used here.

A series of experiments are performed which demonstrate that this technique can successfully remove high frequency oscillations from baroclinic model integrations, even in the presence of topography. It is also shown that the technique produces a consistent and physically reasonable vertical motion field. Other experiments examine the effect of physical parameterizations on the adjusted initial fields.

2 Governing equations

The governing equations are the set of baroclinic primitive equations in sigma co-ordinates: \( \sigma = \) pressure divided by surface pressure. This set of equations, which includes the equations of motion, and the thermodynamic continuity and hydrostatic equations is identical to that given in Section 2 of SD.

It is convenient to introduce a basic state about which we will linearize the equations. This is defined to be a state of rest with a basic state temperature \( T^* \) which is a function of \( \sigma \) only. With this definition, the complete governing equations can be written as follows:
\[ \frac{\partial V}{\partial t} + f k \times V + \nabla(\Phi + RT^* q) = R_V, \quad (1) \]

\[ \frac{\partial T}{\partial t} - \sigma \frac{\partial T^*}{\partial \sigma} - \gamma^*(\hat{D}^\sigma - \hat{D}) = R_T, \quad (2) \]

\[ \frac{\partial q}{\partial t} + \hat{D} = R_q, \quad (3) \]

\[ \sigma \frac{\partial \Phi}{\partial \sigma} = -RT \quad (4) \]

where

\[ R_V = -\xi k \times V - \nabla \frac{V^2}{2} - RT' \nabla q - \sigma \frac{\partial V^2}{2\sigma} - F_V, \quad (5) \]

\[ R_T = -V \cdot \nabla T + \sigma \gamma' + \frac{RT}{c_p} V \cdot \nabla q - \frac{RT'}{c_p} \hat{D} - \frac{RT'}{c_p} \hat{V} \cdot \nabla q \]

\[ + \gamma^* [\hat{V}^\sigma - \hat{V}] \cdot \nabla q + H_T, \quad (6) \]

\[ R_q = -\hat{V} \cdot \nabla q, \quad (7) \]

\( k = \text{unit vector in vertical, } V = \text{horizontal vector wind, } \xi = k \cdot \nabla \times V, \text{ vertical vorticity component, } D = \nabla \cdot V, \text{ horizontal divergence, } q = \text{natural logarithm of surface pressure, } T = \text{temperature, } \Phi = \text{geopotential} \)

\[ \gamma = \frac{RT}{c_p \sigma} - \frac{\partial T}{\partial \sigma}, \text{ static stability,} \quad (8) \]

\[ \hat{F} = \int_\sigma^1 F d\sigma, \hat{F} = \hat{F}_0 = \int_0^1 F d\sigma, \]

\[ \hat{\sigma} = (\sigma - 1)(\hat{D} + \hat{V} \cdot \nabla q) + \hat{D}^\sigma + \hat{V}^\sigma \cdot \nabla q \quad (9) \]

\( f = \text{Coriolis parameter, } R = \text{gas constant for dry air, } c_p = \text{specific heat of dry air at constant pressure, } H_T = \text{thermodynamic forcing, } F_V = \text{frictional forcing, } T^* = T^*(\sigma) = \text{basic state temperature,} \)

\[ \gamma^* = \frac{RT^*}{c_p \sigma} - \frac{\partial T^*}{\partial \sigma} = \text{basic state static stability,} \quad (10) \]

\( T' = T - T^*, \text{ perturbation temperature, } \gamma' = \gamma - \gamma^* \text{ perturbation static stability.} \)

Equations (1)-(4) together with the vertical boundary conditions \( \sigma = 0 \) at \( \sigma = 0, 1 \) constitute the governing equations of the model. The appropriate moisture equation for this model (eqn. 7 of Daley et al., 1976) does not explicitly enter the initialization procedure and thus does not appear here. However, there is a latent heat contribution to \( H_T \) whenever saturation occurs and this effect is included in the initialization procedure.
Non-Linear Normal Mode Initialization

We will also introduce 2 auxiliary variables $W$, $P$ which will be useful in determining the eigenstructure of the model,

$$W = \dot{D}^N - \dot{D}, \text{ or } D + \frac{\partial W}{\partial \sigma} = 0,$$

$$P = \Phi + RT^* q.$$  

3 Construction of the normal modes of the linearized equations

The first step in the initialization process is to determine the free normal modes of the linearized form of the governing equations, over the domain of interest. With $R_v, R_T, R_q = 0$, (1)–(4) can be written as follows, in spherical polar coordinates

$$\frac{\partial u}{\partial t} - 2\Omega \sin \phi v + \frac{1}{a \cos \phi} \frac{\partial P}{\partial \lambda} = 0,$$

$$\frac{\partial v}{\partial t} + 2\Omega \sin \phi u + \frac{1}{a} \frac{\partial P}{\partial \phi} = 0,$$

$$\frac{\partial^2 P}{\partial t \partial \sigma} + \frac{R_T^*}{\sigma} W = 0,$$

$$\frac{\partial P_s}{\partial t} - RT_s^* W_s = 0,$$

where $\lambda = \text{longitude}$, $\phi = \text{latitude}$, $a = \text{Earth's radius}$, $T_s^* = T_s^*|_{\sigma=1}$, $W_s = W|_{\sigma=1}$, $P_s = P|_{\sigma=1}$, $u,v = \text{zonal, meridional velocity components}$, $\Omega = \text{angular velocity of Earth}$.

The following diagnostic relation completes the set

$$\frac{\partial W}{\partial \sigma} = \frac{1}{a \cos \phi} \left[ \frac{\partial u}{\partial \lambda} + \frac{\partial}{\partial \phi} (v \cos \phi) \right].$$

We assume an expansion of the dependent variables $u, v, W, P$ as follows:

$$\begin{bmatrix} u \\ v \\ W \\ P \end{bmatrix} = \begin{bmatrix} u^m \\ v^m \\ iW^m \\ 2\Omega P^m \end{bmatrix} \exp i(m\lambda - 2\Omega \sigma t),$$

where $u^m, v^m, W^m, P^m$ are functions of $(\phi, \sigma)$, $m$ is the zonal wavenumber, $\alpha$ is a non-dimensionalized frequency.

The substitution of (18) into (13)–(17) yields an equation in one variable $W^m$,

$$\frac{\partial^2 W^m}{\partial \sigma^2} - \frac{R_T^*}{4\Omega^2 a^2 \sigma} H^m(W^m) = 0,$$
where

\[ H^m = \left\{ \frac{1}{\cos \phi} \frac{\partial}{\partial \phi} \frac{\cos \phi}{\cos \phi (\alpha^2 - \sin^2 \phi)} \frac{\partial}{\partial \phi} + \frac{1}{(\alpha^2 - \sin^2 \phi)} \times \left[ \frac{m (\alpha^2 + \sin^2 \phi)}{\alpha (\alpha^2 - \sin^2 \phi)} - \frac{m^2}{\cos^2 \phi} \right] \right\} \]  

(20)

is the well-known horizontal structure operator (e.g. Kasahara, 1976).

We assume a separation of variables in (19), \( W^m(\sigma, \phi) = Z(\sigma)X^m(\phi) \) which leads to a horizontal structure equation

\[ H^m(X^m) + \frac{4\Omega^2 a^2}{gh} X^m = 0, \]  

(21)

and a vertical structure equation

\[ \frac{\partial^2 Z}{\partial \sigma^2} + \frac{R*}{\sigma gh} Z = 0, \]  

(22)

where \( h \) is known as the equivalent depth, \( g \) is the gravitational constant, and \(-1/gh\) is the separation constant.

The horizontal structure equation (21) is well-known and has Hough Functions (Longuet-Higgins, 1968) for equivalent depth \( h \) as eigensolutions. The Hough Functions can be divided into 3 classes on the basis of frequency: Rossby modes, westward gravity modes and eastward gravity modes. The horizontal eigenmodes of the model can, in principle, be obtained from (21), but it is much more convenient to use the equivalent procedure discussed in D78.

Of more interest is the determination of the eigenmodes of the vertical structure equation (22), which being second order in \( \sigma \), requires two boundary conditions to properly specify it. At the top, we have from the definition of \( W \) (11) that

\[ W = 0 \quad \text{at} \quad \sigma = 0. \]  

(23)

The bottom boundary condition (at \( \sigma = 1 \)) can be determined as follows. By applying (13), (14) and (17) at \( \sigma = 1 \) together with (16), an equation for \( W^m \big|_{\sigma=1} \) can be obtained.

\[ \frac{\partial W^m}{\partial \sigma} + \frac{RT^*}{4\Omega^2 a^2} H^m(W^m) = 0 \quad \text{at} \quad \sigma = 1. \]

Assuming the same separation of variables as before, yields the horizontal structure equation (21) plus the lower boundary condition for the vertical structure equation (22)

\[ \frac{\partial Z}{\partial \sigma} - \frac{RT^*}{gh} Z = 0 \quad \text{at} \quad \sigma = 1. \]  

(24)

There is a difficulty with the above analysis when the surface topography
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is non-zero. In this case, the assumed basic state (no motion, \( T^* \) a function of \( \sigma \) only) is inconsistent, except in the special case of an isothermal atmosphere (\( T^* \) independent of \( \sigma \)). We will return to this point at the end of Section 6.

Equation (22) subject to boundary conditions (23) and (24) is not a Sturm-Liouville system because the eigenvalue appears in the lower boundary condition. However, it can be shown that any two eigensolutions \( Z'(\sigma), Z^k(\sigma) \) of (22) must satisfy

\[
\int_0^1 \frac{\gamma^*}{\sigma} Z' Z^k d\sigma + T^*_a Z'(1)Z^k(1) = 0, \quad \ell \neq k, \quad (25)
\]

or the alternative form

\[
\int_0^1 \frac{\partial Z'}{\partial \sigma} \frac{\partial Z^k}{\partial \sigma} d\sigma = 0, \quad \ell \neq k. \quad (26)
\]

It might be noted from equations (13)–(17) that the vertical structure functions \( J_0 \) are the appropriate vertical expansion functions for the variables \( u, v, P \).

4 Vertical structure functions of the vertically discretized model equations

The next step is to find the vertical eigenmodes of (22) consistent with the vertical discretization assumed in SD. Consequently, we assume that over the vertical domain \( 0 \leq \sigma \leq 1 \) there are \( N + 1 \) discrete levels \( \sigma_i \) with \( \sigma_0 = 0 \) and \( \sigma_N = 1 \). We then introduce a series of linear finite elements \( e^i(\sigma) \) (also known as Chapeau functions), which are defined as

\[
e^i(\sigma) = (\sigma - \sigma_{i-1})/(\sigma_i - \sigma_{i-1}) \quad \text{for} \quad \sigma \in [\sigma_{i-1}, \sigma_i],
\]

\[
e^i(\sigma) = (\sigma_{i+1} - \sigma)/(\sigma_{i+1} - \sigma_i) \quad \text{for} \quad \sigma \in [\sigma_i, \sigma_{i+1}],
\]

\[
e^i(\sigma) = 0 \quad \text{otherwise}. \quad (27)
\]

The dependent variable \( Z(\sigma) \) is then expanded in a finite series of these basis functions

\[
Z(\sigma) = \sum_{i=0}^N Z_i e^i(\sigma) \quad (28)
\]

Equation (22) is vertically discretized by a Galerkin procedure, that is, the equation is multiplied by an arbitrary basis function \( e^k(\sigma) \) and integrated over the domain for each of the basis functions in turn. This procedure is formally equivalent to that discussed in the Appendix of SD. Thus (22) can be written as the set of equations

\[
\int_0^1 \frac{\partial^2 Z}{\partial \sigma^2} e^k d\sigma + \frac{R}{gh} \int_0^1 \frac{\gamma^*Z}{\sigma} e^k d\sigma = 0, \quad 1 \leq k \leq N.
\]

If we then integrate the first term by parts and make use of the vertical
boundary conditions (23) and (24), the equation becomes

$$\int_0^1 \frac{\partial Z}{\partial \sigma} \frac{\partial e^k}{\partial \sigma} d\sigma - \frac{R}{g h} \int_0^1 \frac{\gamma* Z}{\sigma} e^k d\sigma - \frac{R T*}{g h} Z e^k \bigg|_{\sigma=1} = 0.$$  

Expanding $Z(\sigma)$ in the basis functions (28) and making use of the definition of $\gamma* (10)$ then yields

$$\sum_{i=1}^{N} Z_i a_{ki} - \frac{1}{gh} \sum_{i=1}^{N} Z_i b_{ki} = 0, \quad 1 \leq k \leq N,$$

where

$$a_{ki} = \int_0^1 \frac{\partial e^l}{\partial \sigma} \frac{\partial e^k}{\partial \sigma} d\sigma$$

$$b_{ki} = R \left\{ \sum_{j=1}^{N} T_j^* \int_0^1 \left[ \frac{Re^j}{\sigma c_p} - \frac{\partial e^l}{\partial \sigma} \right] e^k d\sigma + T_N^* \delta_N \delta_N^k \right\}$$

$$T^*(\sigma) = \sum_{j=1}^{N} T_j^* e^l,$$

$$e^l|_{\sigma=1} = \delta_N^k \text{ (Kronecker delta)},$$

$$T_N^* = T_2^* = T^*|_{\sigma=1}.$$

Equation (29) can then be written as a standard algebraic eigenvalue problem,

$$(B^{-1}A - h^{-1}g^{-1}I)Z = 0$$

where $B, A$ are symmetric, tri-diagonal matrices whose elements are $b_{ki}, a_{ki}$. $I$ is the identity matrix, $Z$ is a column vector whose elements are $Z_i, 1 \leq i \leq N$.

We will assume $Z^*(\sigma)$ to be the vertical eigenfunctions of (30) corresponding to the eigenvalue $1/gh^l, 1 \leq l \leq N$. They are obtained from (30) by standard algebraic eigenvalue procedures. $h^l$ is known as the equivalent depth corresponding to the vertical eigenvector $Z^l$. Figure 1 shows the eigenvectors for an 8-level version of the model with $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8 = 0.1, 0.2, 0.3, 0.5, 0.7, 0.8, 0.9, 1.0$, respectively. The basic state temperature field $T^*$ is obtained from the NACA Standard Atmosphere with a surface pressure of 1000 mb. The eigenfunctions are ordered by decreasing equivalent depth. The appropriate equivalent depths corresponding to the eigenvectors in Fig. 1 are given in Table 1.

The first eigenmode is the external mode and the remaining modes are internal modes. It might be noted that Fig. 1 shows the vertical eigenmodes of $\mathcal{W}$ (defined in 11) which are equal to zero at $\sigma = 0$.

Because of the properties of the matrices $B$ and $A$ in (30), its eigenvectors satisfy the following orthogonality condition

$$Z^T B Z^k = 0, \quad l \neq k,$$
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![Vertical structure functions of the model corresponding to the equivalent depths in Table 1.](image)

Fig 1 Vertical structure functions of the model corresponding to the equivalent depths in Table 1.

<table>
<thead>
<tr>
<th>Table 1. Equivalent depths in m.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h^1$ 9614.70</td>
</tr>
<tr>
<td>$h^2$ 2463.27</td>
</tr>
<tr>
<td>$h^3$ 349.22</td>
</tr>
<tr>
<td>$h^4$ 100.77</td>
</tr>
</tbody>
</table>

where (T) indicates transpose. Equation (31) is the consistent discrete analogue of (25).

This completes the determination of the eigenstructure of the linearized model. The vertical structure functions $Z^e(\sigma)$ and the corresponding equivalent depths $h^e$ can be obtained from (30). The horizontal structure functions for each equivalent depth $h^e$ can be obtained in principle from (21), although in practice a procedure identical to that described in D78 is used.

5 Non-linear normal mode initialization

Having obtained the normal modes of equations (1)–(4) with $R_v, R_T, R_q = 0$, the next step is to use them to initialize the model. The goal is to adjust the initial data in such a way that the model integration of these full equations does not contain high frequency oscillations.

This goal is accomplished by first requiring that the undesirable high frequency modes be identified and that the initial data be projected on these modes. As discussed in D78, linear normal mode initialization is performed by simply removing these modes from the initial data. This, unfortunately, is not sufficient to remove high frequency oscillations from a subsequent integration of the full non-linear equations, because the non-linear terms act as forcing terms to re-generate the high frequency oscillations.

However, as originally demonstrated by Machenhauer (1977) and Baer (1977), if the non-linear terms themselves do not change rapidly with time,
then it is possible to produce a better initialization by expanding the non-linear terms \( R_V, R_T, R_q \) into the normal modes of the linearized equations and then adjusting the original data such that the time tendencies of the undesirable modes are initially zero.

The methodology for performing the non-linear normal mode adjustment for the shallow water equations is outlined in D78, Section 3. The present discussion will exactly parallel that methodology except that the application will be to the baroclinic equations (1)–(4). Note that the variational generalization of the method discussed in Section 4 of D78, will not be considered here.

The application of the non-linear normal mode procedure to the baroclinic equations can be broken down into several steps, assuming that the eigenstructure (horizontal and vertical) of the linearized model has been determined and the undesirable modes identified. We must first vertically decompose the dependent variables of the model and the non-linear (and forcing) terms, using the vertical structure functions derived earlier. This will produce, for each equivalent depth \( h' \), a set of equations analogous to the shallow water equations, (Taylor, 1936). Then for each equivalent depth \( h' \), we will perform a non-linear adjustment to these analog shallow-water equations, exactly as in D78 Section 3. This will produce adjustments which are then vertically transformed back to real space, to produce new initial values of the dependent variables. As in D78, we then re-calculate the non-linear terms and repeat the whole process until convergence.

We will now discuss each step of the procedure in more detail.

**Step a Vertical Projection of the Model Dependent Variables and Non-Linear (and Forcing) Terms.**

The actual form of the governing equations (1)–(4) integrated in the model described in SD is obtained by differentiating equation (1) horizontally and vertically and applying (4) and (11) to give the following equivalent form:

\[
\frac{\partial \xi}{\partial t} + \nabla \cdot f \nabla = k \cdot \nabla \times R_V = R_\xi
\]

\[
\frac{\partial^3 W}{\partial \sigma^2 \partial t} + k \cdot \nabla \times f \frac{\partial V}{\partial \sigma} + RV^2 \left( \frac{T'}{\sigma} - \frac{\partial T^*}{\partial \sigma} q \right) = \frac{\partial}{\partial \sigma} \nabla \cdot R_V = -\frac{\partial R_p}{\partial \sigma},
\]

\[
\frac{\partial T}{\partial t} - \sigma \frac{\partial T^*}{\partial \sigma} \frac{\partial q}{\partial t} - \gamma^* W = R_T,
\]

\[
\frac{\partial q}{\partial t} - W_s = R_q.
\]

The history-carrying variables of the model are \( \xi, W, T' \) and \( q \) with \( V \) being obtained from \( \xi \) and \( W \) by the definitions in Section 2.

The correct method for vertically decomposing these equations must rigorously follow the vertical discretization procedures utilized in the model. Since
most of the detail in this vertical decomposition is peculiar to the present model, it has been placed in the Appendix and only the final result will be given here.

We define vertically transformed variables $\xi^\ell(h, \phi, t)$, $D^\ell(h, \phi, t)$, $\Phi^\ell(h, \phi, t)$ for each equivalent depth $h^\ell$, $1 \leq \ell \leq N$ where $\xi^\ell$ is the vertically transformed vorticity variable, $D^\ell$ is the transformed $W$ variable and $\Phi^\ell$ is the transformed mass variable and a tilde ($\sim$) indicates a vertically transformed variable. The transformation of $W$ into $D^\ell$ is the most straightforward because the vertical structure functions $Z^\ell(\sigma)$ obtained from (30) are the vertical structure functions corresponding to $W$.

Thus we can write for a given $(h, \phi, t)$

$$W_i = W(\sigma_i) = \sum_{\ell=1}^{N} D^\ell Z^\ell(\sigma_i), \quad 1 \leq i \leq N,$$

or alternatively using matrix notation

$$W = ED$$

(36)

where $E$ is the matrix of eigenvectors $Z^\ell(\sigma_i)$ of (30), $D$ is the column vector whose elements are given by $D^\ell$, $1 \leq \ell \leq N$, $W$ is the column vector of nodal values of $W(\sigma)$, i.e. the values $W(\sigma_i) = W_i$, $1 \leq i \leq N$.

Given values of $W$ at the nodes of the vertical mesh, $\sigma_i$, $1 \leq i \leq N$, $D^\ell$ can be obtained simply by inverting (36): $D = E^{-1}W$. The corresponding transformations to produce $\xi^\ell$ from $\xi$ and $\Phi^\ell$ from $T'$, $q$ and $\Phi_s$ (surface topography) are more complicated to derive (see Appendix) and they are given below in matrix form.

$$AE\xi = C\xi,$$

(37)

$$AE\Phi = CP,$$

(38)

where $\xi$, $P$ are column vectors with elements $\xi(\sigma_i)$, $P(\sigma_i)$, $\xi$, $\Phi$ are column vectors with elements $\xi^\ell$, $\Phi^\ell$, $C$ is a matrix whose elements $C_{ki}$ are defined by

$$C_{ki} = \int_{0}^{1} e^k \frac{\partial e^i}{\partial \sigma} d\sigma - \delta_N^k \delta_N^i,$$

(39)

$CP$ can also be written in terms of $T'$, $T^*$, $q$ and $\Phi_s$

$$CP = -R \int_{0}^{1} \left[ \frac{T'}{\sigma} - q \frac{\partial T^*}{\partial \sigma} \right] e^k d\sigma - \delta_N^k [\Phi_s + RT_N^* q].$$

Thus the vertically decomposed vorticity $\xi^\ell$ can be calculated from $\xi$ by (37) and the vertically decomposed mass variable $\Phi^\ell$ can be calculated from $T'$, $q$ and $\Phi_s$ by (38). In a similar fashion the non-linear (and forcing) terms $R_\xi$, $R_D$, $R_T$, $R_q$ in (32)–(35) can be vertically decomposed into $\tilde{R}_\xi^\ell, \tilde{R}_D^\ell, \tilde{R}_T^\ell, \tilde{R}_q^\ell$, $1 \leq \ell \leq N$.

**Step b  Horizontal Decomposition and Adjustment**

With the use of equations (36)–(39) the governing equations (32)–(35) can be
completely decomposed into the following for each equivalent depth $h^\epsilon$, $1 \leq \epsilon \leq N$,

### Equations

\[
\begin{align*}
\frac{\partial \bar{\xi}^\epsilon}{\partial t} + \nabla \cdot f \bar{v}^\epsilon &= \bar{R}_\xi^\epsilon, \\
\frac{\partial \bar{D}^\epsilon}{\partial t} - k \cdot \nabla \times f \bar{v}^\epsilon + \nabla^2 \bar{D}^\epsilon &= \bar{R}_D^\epsilon, \\
\frac{\partial \bar{\Phi}^\epsilon}{\partial t} + gh^\epsilon \bar{D}^\epsilon &= \bar{R}_\phi^\epsilon,
\end{align*}
\]

where $\bar{\xi}^\epsilon = k \cdot \nabla \times \bar{v}^\epsilon$, $\bar{D}^\epsilon = \nabla \cdot \bar{v}^\epsilon$.

The linear terms on the left-hand side of (40)–(42) are identical to the linearized shallow-water equations for equivalent depth $h^\epsilon$. The right-hand side terms $\bar{R}_\xi^\epsilon, \bar{R}_D^\epsilon, \bar{R}_\phi^\epsilon$, however, are the vertical projections of the non-linear and forcing terms of the baroclinic equations, instead of being the non-linear terms of the shallow-water equations. It will be noted that the left-hand sides of equations (40)–(42), written in polar spherical coordinates, are equivalent to the linearized shallow-water equations used in D78.

The next step in the procedure, the horizontal decomposition and adjustment, is virtually identical to that of D78 and will not be described in detail. We assume we have already computed the horizontal structure functions (Hough Functions) for equivalent depth $h^\epsilon$ as in D78 and have identified the undesired modes. We then proceed to calculate the amplitude of the undesired modes by projecting $\bar{\xi}^\epsilon, \bar{D}^\epsilon, \bar{\Phi}^\epsilon$ on these modes. The next phase is to project $\bar{R}_\xi^\epsilon, \bar{R}_D^\epsilon$ and $\bar{R}_\phi^\epsilon$ on the time tendencies of these undesired modes. The last stage is to calculate a new initial amplitude for these modes, such as to zero their initial time tendencies. This produces a set of adjustments $\Delta \bar{\xi}^\epsilon, \Delta \bar{D}^\epsilon$ and $\Delta \bar{\Phi}^\epsilon$ for each equivalent depth $h^\epsilon$.

### Step c Reverse Vertical Transform

The next step is to vertically transform the adjustments $\Delta \bar{\xi}^\epsilon, \Delta \bar{D}^\epsilon, \Delta \bar{\Phi}^\epsilon$ back to real space to find the adjustments to the raw fields, i.e. to find $\Delta \xi, \Delta \bar{W}, \Delta T', \Delta q$.

$\Delta \bar{W}$ can be produced from (36) and $\Delta \xi$ from (37) in a straightforward manner. Obtaining $\Delta T'$, $\Delta q$ from $\Delta \bar{\Phi}^\epsilon$ is more complicated. The appropriate form of (38) is

\[
\begin{align*}
\Delta \bar{\Phi}^\epsilon = -RF \Delta T' + R\Delta q CT^* 
\end{align*}
\]

where $\Delta T', T^*$ are column vectors whose elements are $\Delta T'(\sigma_i), T^*(\sigma_i), 1 \leq i \leq N$, and $F$ is the matrix whose elements are

\[
f_{ik} = \int_0^1 \frac{\phi e^i}{\sigma} d\sigma.
\]

For an $N$-level model there are $N$ adjustments $\Delta \bar{\Phi}^\epsilon$, but a total of $N + 1$ real space adjustments ($\Delta T'_i, 1 \leq i \leq N$; plus $\Delta q$). In fact there is a redundancy and $\Delta T'$ and $\Delta q$ can be adjusted in any manner consistent with (43). For example,
Non-Linear Normal Mode Initialization

$\Delta q$ could be set equal to zero, and (43) used to solve for $\Delta T'$. A more consistent method is to note

$$ \Delta \Phi = C \Delta P, $$

where $\Delta P$ is the column vector whose elements are

$$ \Delta P_i = \Delta P(\sigma_i) = \Delta \Phi(\sigma_i) + RT^*(\sigma_i) \Delta q. $$

Thus if $\Delta \Phi$ is known, $\Delta P$ can be obtained from (44). Using the relation $\Delta P|_{\sigma=1} = \Delta P^N = RT_N^* \Delta q$ we can find $\Delta q$. The substitution of $\Delta q$ back into (43) gives $\Delta T'$.

In practice there are certain difficulties with this technique which will be discussed later. Consequently a variational solution to (43) was also devised. We demand that the following variational integral

$$ I(\Delta T', \Delta q, W_T, W_q) = \int_0^1 [\Delta T'(\sigma)]^2 W_T(\sigma)d\sigma + [T_N^* \Delta q]^2 W_q, $$

be minimized subject to the $N$ constraints (43). Here, $W_T(\sigma)$ and $W_q$ are user-supplied weight functions, and $T_N^* = T^*|_{\sigma=1}$ has been included for dimensional consistency. The integral $I$ is then minimized by using Lagrange's Undetermined Multipliers. Details can be seen in Daley (1978a).

**Step d Calculation of New Initial Fields and Iteration Step**

The final step is to adjust the original fields which we will denote by $\xi(0), W(0), T'(0), q(0)$ using the adjustments $\Delta \xi, \Delta W, \Delta T'$ and $\Delta q$. Thus the adjusted fields, which we will denote by $\xi(1), W(1), T'(1), q(1)$ can be obtained simply by adding the original fields and the adjustments, i.e. $\xi(1) = \xi(0) + \Delta \xi$.

As in Machenhauer (1977) and D78 we then use $\xi(1), W(1), T'(1), q(1)$ to calculate new values for $R_\xi, R_D, R_T, R_q$ and repeat the whole process to calculate better balanced fields $\xi(2), W(2), T'(2), q(2)$. This process can then be repeated until the adjusted fields are considered to have converged. It was generally found that 3 passes through this iteration procedure produced adequate convergence (except in a few cases discussed in the next section).

**6 Results**

The practical application of the initialization procedure developed above was found to be fairly straightforward. Expected difficulties with the method in the presence of topography did not materialize.

It was discovered, however, that the adjustment of $\Delta T'$, $\Delta q$ by the use of equation (44) resulted in a rather small adjustment in $\Delta q$ and a rather large adjustment in $\Delta T'$. In some regions this produced a considerable static instability in the lower atmosphere. Consequently, in practice, the variational formulation (45) was preferred to produce $\Delta T'$, $\Delta q$ from $\Delta \Phi$. Naturally, the adjustment then became somewhat dependent on the choice of weights $W_T$ and $W_q$. For all the experiments reported here $W_T(\sigma) = 1$ and $W_q = 0.01$ were chosen and the problem of static instability disappeared.
All experiments were run from the same set of original analyses. These were 10-level polar stereographic constant pressure analyses of geopotential, temperature, winds and dew-point depression for 0000 GMT February 3, 1976, obtained by the multivariate optimum interpolation scheme of Rutherford (1978). The original divergent wind field in these analyses was discarded. The constant pressure data were then vertically and horizontally interpolated to produce $\xi$, $W$, $T'$, and $q$ (plus a moisture variable) on the sigma surfaces of the model. At this point, the non-linear normal mode initialization procedure could be applied. Then, the full baroclinic primitive equation model described in SD was run from these data. The forecasts from the model could then be examined either in pressure co-ordinates (which involved possible interpolation and extrapolation error) or in sigma co-ordinates (in which forecasts might be difficult to interpret).

The model configuration chosen for the experiments had a horizontal resolution of rhomboidal 21 (hemispheric), 8-levels in $\sigma$ (0.1, 0.2, 0.3, 0.5, 0.7, 0.8, 0.9, 1.0) and a time-step of 30 min. Most experiments were run without topography to minimize output problems (the experiments performed with topography are described separately). All other physical effects mentioned in Section 1 were included in both the initialization procedure and the model integration unless otherwise noted. In all cases the parameters in the initialization procedure were set exactly the same as in the model run from the initialized data. For this 8-level model, the appropriate vertical structure functions and equivalent depths are shown in Fig. 1.

In the application of normal mode initialization procedures, the user must decide in advance which modes he considers undesirable and therefore wishes to adjust by the procedure of Section 5. One could establish a criterion based strictly on the frequencies of the eigenmodes, only adjusting those modes whose frequencies exceeded a certain given value.

We have opted for an alternate criterion for deciding which modes will be adjusted. For a given equivalent depth $h'$, $\frac{3}{4}$ of the modes can be interpreted as Rossby modes, $\frac{1}{4}$ as eastward gravity modes and $\frac{1}{4}$ as westward gravity modes. In D78, we simply adjusted all gravity modes from the model integration. In the present study, a similar criterion has been used, i.e. we simply adjust all the gravity modes for equivalent depths greater than a certain critical equivalent depth. Thus, by suitably choosing the critical equivalent depth we can adjust all external gravity modes or all external-plus-first internal gravity modes, etc. This procedure is fairly reasonable for the external mode, where the frequencies of the Rossby and gravity modes are well separated (the mixed Rossby-gravity mode does not appear in the hemispheric case); but becomes less and less justifiable for the higher internal modes.

In the experiments to follow we thus define 5 cases listed below: in case 0 there is no initialization; all the remaining cases correspond to initialization with different critical equivalent depths, e.g. in case 1 all external gravity modes are adjusted.
Non-Linear Normal Mode Initialization / III

Case 0 – no initialization

Adjustment of all gravity modes corresponding to:
  Case 1 – the external mode,
  Case 2 – external + first internal modes,
  Case 3 – external + first 2 internal modes,
  Case 4 – external + first 3 internal modes.

As mentioned in Section 5, Step d, the adjustment procedure involved a non-linear iteration loop. It cannot be assumed, a priori, that this iteration loop will converge. For the 8-level model considered here, convergence could be obtained if the first 6 or less of the vertical modes were adjusted. However, when the adjustment procedure was applied to all 8 vertical modes (in the presence of topography and all the physical parameterizations of the model) convergence was not obtained.

The first and most crucial experiment clearly demonstrates the ability of the present initialization procedure to suppress high frequency oscillations from model integrations. In Fig. 2 we plot values of height at 4 constant pressure levels (100, 300, 500, 1000 mb) for a fixed latitude-longitude during a 24-h model integration, with and without initialization. The experiment was run without topography, and the output is plotted every 2 h.

The curves plotted in Fig. 2 correspond to the five cases listed above. Since many of the cases give virtually the same results, we have also adopted the following convention in the interest of clarity: if the curve corresponding to a particular case number is not plotted, then it can be assumed that the curve is virtually identical to that of the highest case number plotted. For example, in Fig. 2 at 500 mb, only cases 0, 1 and 2 are plotted, implying, for 500 mb, that cases 3, 4, etc., are virtually identical to case 2, i.e. the adjustment of gravity modes with equivalent depth smaller than that of the first internal mode produces virtually no change.

Figure 2 demonstrates that for no initialization (case 0) there are high amplitude external gravity waves present in the model integration. It might be noted that this point (13.4°N, 225°E) is in the tropics where gravity wave oscillations in the model have been known to be particularly serious. An examination of case 1 in Fig. 2 shows that the adjustment of only the external gravity waves in the initialization procedure is sufficient to remove most of the high frequency oscillations from the forecast geopotential. In fact, for the troposphere, it is hardly necessary to adjust any of the internal modes at all in order to remove high frequency oscillations. In the stratosphere (100 mb), however, it would seem that the uninitialized integration (case 0) has substantial amplitude internal gravity modes which the present procedure is capable of removing (case 3). Stratospheric internal gravity waves have also been noted by Dey (1978) in the NMC 8-level global model.

In Fig. 3a is plotted the initial root-mean-square zonally averaged height adjustments as a function of latitude for the same 4 pressure levels. These
Fig. 2 Model forecast height (m) at four constant pressure levels (100, 300, 500, 1000 mb) every two h at the point (13.4°N, 225°E) (no topography). Case: 0 ——; 1 —; 2 ....... 3 – – – ; 4 x —— x.

adjustments correspond to cases 1 to 4 defined earlier and there is no topography. Fig. 3b shows the same results for the root-mean-square vector wind adjustment (rotational component). It can be seen that most of the height adjustment occurs in cases 1 and 2 and there is little adjustment after that. In the case of the rotational part of the wind, very little adjustment occurs until case 3. This is consistent with equation (4.13) of D78.

In the case of the height adjustment, the results in Fig. 3a are fairly comparable to those found for the barotropic adjustment of D78. The wind adjustment (Fig. 3b), however, for cases 3 and 4 is somewhat larger than in D78. Neglecting the peaks at the North Pole (small sample size) we see that the wind adjustments are as large as 4 m s⁻¹, which is not completely negligible. To put these wind adjustments into perspective they might be compared with
the analysis errors in the raw analyses. These errors are, of course, not known, but estimates have been obtained by Larsen et al. (1977). A comparison of Fig. 3 with these estimated analysis errors suggests that in the most severe case (4) the initial adjustment is somewhat smaller than the estimated analysis error.

What remains unclear is whether these large wind adjustments in cases 3 and 4 are characteristics of non-linear normal mode initialization or are due to peculiarities of the present experimental set-up. It is obvious that there are many factors which will influence the adjustments that occur. The particular horizontal and vertical discretization and the physical parameterization in the vertical initialized model will be important. In fact it was found that the wind adjustments of Fig. 3b were very sensitive to the number of vertical degrees of freedom. One might also expect that such factors as the type of vertical interpolation (pressure to sigma) or the weights \( W_T(\sigma) \), \( W_q \) might affect the

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**Fig. 3**  
Fig. 3 a - Initial root-mean-square adjustment in the height field (m); b - vector rotational wind (m s\(^{-1}\)) as a function of latitude for the same 4 constant pressure levels as Fig. 2 (no topography). Legend for curves is given in Fig. 2.
adjustments, particularly in the higher internal modes. In addition, the particular data and analysis procedure used to obtain the raw initial analyses might also be significant. For example, in regions of low data density, the initial analyses consist essentially of trial fields obtained from a numerical forecast model. In the present case, this trial field model was a grid-point polar stereographic primitive equation model similar to that of Robert et al. (1972).

The adjustments in these data-sparse regions could conceivably be quite different if the present model (SD) were used instead to produce the trial fields for the initial raw analyses.

In Fig. 4a is plotted the hemispheric root-mean-square height difference between the uninitialized model (case 0) and the initialized model cases (1–4) for every 6 h for 1 day. The same 4 constant pressure levels (100, 300, 500, 1000 mb) are shown and there is no topography. Figure 4b shows the similar plots for the hemispheric root-mean-square vector wind difference (rotational component). Both Figs 4a and 4b indicate that in the troposphere the differences between the uninitialized and the initialized runs decay with time. This is not surprising because the dissipative mechanisms in the model will gradually tend to damp out high frequency oscillations in the uninitialized run. The results in the stratosphere (100 mb) are again anomalous.

The next series of figures demonstrates the initialization procedure’s ability to produce consistent and physically reasonable vertical motion fields. Figs 5a
and b show the initial mean sea-level pressure (and associated frontal structure) and the corresponding 850-mb vertical motion field ($\omega = \text{total time derivative of pressure}$) for case 4, over northeastern North America. (This again is for a model with no topography.) In this region there is an intense cyclone centred over central Quebec with associated troughs over the Great Lakes and along the Eastern Seaboard. There is an associated vertical motion area ($\omega < 0$) in the northeast quadrant of the low and smaller areas of upward motion (marked with U) ahead of the two troughs mentioned above. There are areas of downward vertical motion (marked with D) ahead of each of the ridges in 5a. The correspondence between Figs 5a and 5b is in accord with classical theory and synoptic experience. The intensities of the vertical motions (mb h$^{-1}$) also seem reasonable.

Fig. 6 vertical profiles of vertical motion are plotted at 2-h intervals at the point marked with an X (Labrador Coast) in Fig. 5b. Figure 6a shows the vertical motion profiles for the uninitialized model (case 0) and 6b for the initialized model (case 4). The first panel of Fig. 6b also shows the initial vertical motion profiles for cases 2 and 3. In examining Fig. 6a we see that the initial virtually zero vertical motion of the uninitialized model rapidly becomes large and negative (upward), overshoots at 4 h and settles down by 12 h to a double-peaked structure. In Fig. 6b, case 4 starts out with a very reasonable large and negative vertical motion, which at 12 h becomes very similar to case 0. It is evident from Fig. 6b panel 1, that the initial vertical motion of case 2 (and therefore case 1) is inadequate, while that of case 3 is marginal.

Fig. 7 is a plot of the rainfall rate (mm h$^{-1}$) for the same point as a function of time for 24 h. Two cases are shown – 0 and 4. Case 0 clearly demonstrates a common problem of uninitialized models, a lag of about 6 h before a reason-
able rainfall rate is achieved and high frequency oscillations in the rainfall rate. Case 4, with a reasonable initial vertical motion shows rainfall rates which are free of the high frequency oscillations, but shows only a small improvement in the initial rainfall rates, probably due to inadequacies in the initial moisture analyses. The use of statistical analysis procedures for analyzing the moisture variable used in this model (temperature minus dew-
point temperature) tends to reduce the closeness to saturation in regions which are saturated or nearly saturated. Consequently, even though the initialized model (case 4) has a substantial upward motion at this point, the model atmosphere may not be properly saturated so that it takes several hours to build up a reasonable rainfall rate.

Fig. 8 shows the vertically integrated hemispheric root-mean-square divergence as a function of time for 5 cases. Cases 0, 1, 2 all have insufficient initial divergence, and then there is an overshoot at 4–6 h. The initial divergence in case 3 is marginal, but in case 4 is reasonable; all cases converge by 14 h, consistent with Fig. 6. We note that the time behaviour of the rms divergence in the initialized baroclinic model shown in Fig. 8 is very different from that of the initialized barotropic model discussed in D78. In the latter, the rms divergence was virtually time-invariant and had a relative level much lower than that in the uninitialized baroclinic model.

Figure 9 attempts to show the effect of surface drag on the initialized wind fields. One would expect that a model initialized with surface drag would show a turning of the wind in the surface layers. Unfortunately, not having the facility for directly plotting wind vectors, we have attempted to illustrate this effect indirectly by plotting the surface streamfunction and divergence fields. It would be expected that a model initialized with/without surface drag would have lower/higher surface winds and more/less low level convergence into the centres of cyclones. Figures 9a and b show the streamfunction at $\sigma = 1$, for the initialized no-drag case (corresponding to the mean sea level pressure map shown in Fig. 5a) and the surface drag case. Figures 9c and d show the divergence (D) and convergence (C) at $\sigma = 1$ for the no-drag and drag cases. The expected effect on the low level divergence field is certainly apparent in Figs 9c and 9d. The effect on the surface rotational wind field (Figs 9a and b) is very small, but in the right direction. It was found that 6 vertical modes had to be adjusted before there was an appreciable response to the inclusion or non-inclusion of surface drag. The reason for this can be found by examining the model's surface-drag formulation which essentially has a vertical profile which is zero everywhere, except at $\sigma = 1$. An examination of Fig. 1, would
suggest that it would require the modes with small equivalent depth to respond to this type of forcing function.

The last experiments performed examined the effect of topography on the initialization process. Figure 10a is a plot of 500-mb height as a function of time for the same point (13.4°N, 225.0°E) as in Fig. 2, except that topography has been included in the initialization and model integration. In these experiments a basic state ($T^*(\sigma)$, no motion), as in Section 3, was assumed for the initialization process. The results are very similar to the 500-mb case in Fig. 2. The results for 100, 300, 1000 mb (not shown) are also similar to their counterparts in Fig. 2. Since the point (13.4°N, 225.0°E) is not in the vicinity of high mountains, we have plotted in Fig. 10b a similar curve for the point (72.8°N, 315.0°E) which is directly over the Greenland ice cap. The results of Fig. 10
suggest that even in the presence of topography the initialization procedure is capable of completely removing high frequency oscillations from model integrations. The initial zonally-averaged adjustments for the height and wind fields in the presence of topography are very similar to Fig. 3 and are not shown here.

It was expected that in the presence of topography, the non-linear adjustment scheme would cause large changes in the analyses, particularly over the Himalayas in the vicinity of the tropopause. In fact, at 100 mb, in this region, the maximum wind adjustment was found to be 10 m s$^{-1}$ and the maximum height adjustment was found to be 160 m. These adjustments are of the same order as the vertical interpolation errors (pressure to sigma) determined by Sundquist (1976) for the same region.

At the end of Section 3 it was mentioned that in the case $\Phi_S \neq 0$ the basic state $(T^*(\sigma), \text{no motion})$ assumed for the initialization process, was inconsistent. This difficulty did not arise, however, if $T^*$ were assumed independent of sigma. Consequently, a topographic experiment (not shown) was performed using an isothermal basic state for the initialization process. It was found that in this case, the initially adjusted fields and the time behaviour of the subsequent model integration were very similar to the case where $T^*$ was allowed to vary with sigma.

It might be asked if the initialization procedure can produce initial vertical motion fields consistent with the inclusion of topography, i.e. will the ini-

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Fig. 10  a - Model forecast height (m) at 500 mb for the same point as Fig. 2, except with topography. b - as (a) except for point (72.8°N, 315.0°E). Legend for curves is given in Fig. 2.
Fig. 11  a - Initial zonally averaged 500-mb height (dam) over Western North America; b - model topography (dam) in the same region; c - initial 850-mb vertical motion field (mb h⁻¹) consistent with (a, b). Upward vertical motion is indicated by U and downward by D.

tialized vertical motion be upward on the upstream side of mountain barriers and downward on the downstream side? Since the mountain-induced vertical motion is generally smaller and somewhat hidden by the synoptically-induced vertical motion, we decided to use simplified initial conditions to investigate this effect. We took the original analyses on constant pressure surfaces and zonally averaged them. We then interpolated them to the sigma surfaces of the model (in the presence of topography) and performed an initialization (case 4). We then examined the output (interpolated back to pressure surfaces) in the vicinity of the Rocky Mountain Chain, which is perpendicular to the zonally averaged flow.

Figure 11a shows the 500-mb height field (dam) over western North America, demonstrating the zonal averaging of the fields. Figure 11b shows
the model topography (dam) in this region. Figure 11c shows the resulting 850-mb vertical motion field (\( \omega \) in mb h\(^{-1}\)) with upward vertical motion (U) on the west side (upstream) and downward vertical motion (D) on the east side (downstream) of the Rocky Mountain barrier.

We conclude the Results Section with a remark on the efficiency of this initialization procedure. As in D78 most of the computing time is taken up in calculating the non-linear and forcing terms \((R_x, R_D, R_T, R_q)\). Since acceptable convergence is obtained in 3 scans, we can say that the computation time is roughly equivalent to 3–4 time steps of the model.

### 7 Summary and conclusions

The non-linear normal mode initialization technique of Machenhauer (1977) and Baer (1977) has been successfully applied to a baroclinic primitive equations forecast model. A series of experiments demonstrated that the technique was capable of completely removing high frequency oscillations from model integrations, even in the presence of topography. In addition, the procedure produced a consistent initial vertical motion field. It was also shown that the response of the procedure to certain physical parameterizations in the model, was physically reasonable.

Certain features of the experiment were slightly disquieting – such as the anomalous behaviour in the stratosphere and the relatively large wind adjustment for the higher internal modes. It is possible, however, that these effects reveal more about inadequacies of the model, than of the initialization procedure.

More development work lies ahead before the non-linear normal mode initialization procedures can be used in practice. It would be desirable to put the procedure into a variational context as in D78 and to make a more careful selection of the modes to be suppressed. There is also a requirement for vigorous testing of the procedure with various real data sets to ensure that its use does not cause deterioration in the model forecast.

Since the analysis-prognosis system used in this experiment is designed to produce good quality forecasts over Canada, both the analyses (Rutherford, 1978) and the forecast model (SD) are deficient in the tropical region. Consequently, the question of the applicability of the non-linear normal model initialization procedure in the tropical regions has not been resolved here.

Obtaining the normal modes of a numerical model would seem to be a worthwhile investment. Once a non-linear normal mode procedure has been developed, any improvements in the model automatically become part of the initialization procedure. In addition, the normal modes themselves can be used as a diagnostic tool to monitor model behaviour.

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Appendix

The Derivation of the Vertical Transform Matrices for Vorticity and Mass

The set of baroclinic equations (32)–(35) used in the model can be reduced to a set of analogue shallow-water equations (40)–(42) for each equivalent depth $h^e$, $1 \leq \ell \leq N$ by suitable matrix transformations, of the dependent variables $\xi$, $W$, $T'$, and $q$ and the non-linear terms $R_\xi$, $R_p$, $R_T$ and $R_q$. These matrix transforms (36)–(39) must be consistent with the vertical discretization used in the model. The derivation of all these matrix relations is tedious, so we will merely indicate the procedure followed by considering the vertical discretization of the divergence equation (33). This will yield the vertical transform matrices for vorticity (37), mass (38) and $R_D$.

Equation (33) can be written

$$\frac{\partial^3 W}{\partial \sigma^2 \partial t} + k \cdot \nabla \times f \frac{\partial V}{\partial \sigma} - RV^2 \frac{\partial P}{\partial \sigma} = -R_D, \quad (A-1)$$

where $P$ is defined by equation (12).

As in SD we now apply a Galerkin procedure to (A1), by multiplying through by $e^k(\sigma)$ and integrating over $\sigma$.

$$\int_0^1 \left[ \frac{\partial^3 W}{\partial \sigma^2 \partial t} + k \cdot \nabla \times f \frac{\partial V}{\partial \sigma} - RV^2 \frac{\partial P}{\partial \sigma} + \frac{\partial R_D}{\partial \sigma} \right] e^k d\sigma = 0 \quad (A-2)$$

for each $k$, $1 \leq k \leq N$.

As in SD we integrate the first term by parts

$$\int_0^1 \frac{\partial^3 W}{\partial \sigma^2 \partial t} e^k d\sigma = -\int_0^1 \frac{\partial^2 W}{\partial \sigma \partial t} \frac{\partial e^k}{\partial \sigma} d\sigma + \frac{\partial^2 W}{\partial \sigma \partial t} e^k \bigg|_{\sigma=1}$$

We note that an expression for

$$\frac{\partial^2 W}{\partial \sigma \partial t} \bigg|_{\sigma=1}$$

can be derived by taking the divergence of (1) and applying it at $\sigma = 1$.

$$\frac{\partial^2 W}{\partial \sigma \partial t} + k \cdot \nabla \times f \nabla - \nabla^2 P + R_{pl}\bigg|_{\sigma=1} = 0 \quad (A-3)$$

Inserting (A3) into (A2) and noting that $e^k(\sigma)\big|_{\sigma=1} = \delta_N^k$ we find

$$-\int_0^1 \frac{\partial^2 W}{\partial \sigma \partial t} \frac{\partial e^k}{\partial \sigma} d\sigma + \int_0^1 \left[ k \cdot \nabla \times f \frac{\partial V}{\partial \sigma} - \nabla^2 \frac{\partial P}{\partial \sigma} + \frac{\partial R_D}{\partial \sigma} \right] e^k d\sigma$$

$$- \delta_N^k [k \cdot \nabla \times f V - P + R_D] \bigg|_{\sigma=1} = 0. \quad (A-4)$$
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Expanding $W$, $V$, $P$, $R_D$ in terms of Chapeau functions, (27)-(28), for example,
\[ W = \sum_{i=1}^{N} W_i e^i, \]
we can rewrite (A4) as
\[ -A \frac{\partial W}{\partial t} + C[k \cdot V \times fV - V^2P + R_D] = 0, \quad (A-5) \]
where $V$ is a column vector whose elements are $V(\sigma_i)$, $1 \leq i \leq N$, $P$ is defined in (38), $A$ is defined by (29), $C$ is defined by (39), $R_D$ is a column vector whose elements are $R_D(\sigma_i)$, $1 \leq i \leq N$. Noting that $W = ED$, $A(5)$ can be written
\[ \frac{\partial D}{\partial t} - k \cdot V \times \tilde{V} + \nabla^2 \tilde{\Phi} = \tilde{R}_D \quad (A-6) \]
where $\tilde{V}$ is the column vector defined by $\tilde{\xi} = k \cdot \nabla \times \tilde{V}$, $\tilde{R}_D$ is the column vector whose elements are $\tilde{R}_D e^i$, and $\tilde{\Phi}$ satisfies (38).

We note that (A6) is exactly the same as (41), and we have thus effectively de-coupled the model’s divergence equation (33) using the model’s own vertical discretization.

The remaining model equations (32), (34) and (35) can be vertically de-coupled to produce the analogue shallow-water equations (40 and 42) by exactly the same procedure.

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